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The objectives of this project were (1) to prove convergence theorems for probability-one homotopy methods applied to H^2 and combined H^2/H^∞ optimal model order reduction and controller synthesis problems, and (2) to develop a robust, fixed-structure MATLAB toolbox. This report consists of a paper on convergence theory for homotopy control algorithms, and a user's guide for a MATLAB toolbox. The toolbox is available on the World Wide Web at URL **http://www.cs.vt.edu/~ltw/toolbox/.**

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FINAL TECHNICAL REPORT FOR AFOSR GRANT F49620-96-1-0089

ALGORITHMS AND SOFTWARE FOR COMBINED H^2/H^∞ CONTROL

Period: 4/1/96 – 3/31/97

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April 26, 1997

Objectives.

The objectives of this project were (1) to prove convergence theorems for probability-one homotopy methods applied to H^2 and combined H^2/H^∞ optimal model order reduction and controller synthesis problems, (2) to develop a robust, fixed-structure MATLAB toolbox, (3) and to extend HOMPACK to deal with bifurcation curve tracking.

Accomplishments/new findings.

For three different formulations of the H^2 optimal model order reduction problem (optimal projection equations, input normal form parametrization, and Ly form parametrization), convergence theorems for globally convergent probability-one homotopy algorithms have been proved. Several counterexamples were also developed, showing that the results are sharp. These results complete convergence theory for H^2 optimal model order reduction homotopies. Some progress toward homotopy convergence theory for combined H^2/H^∞ model order reduction and controller synthesis was also made. This work is contained in an *Automatica* paper under review and in Yuan Wang's Ph.D. thesis. The current version of the *Automatica* paper is attached to this report.

The *robust, fixed-structure MATLAB toolbox* can be used to synthesize fixed-structure controllers that are optimal with respect to a given performance measure, and at the same time satisfy stability and robustness constraints. The toolbox can handle centralized or decentralized compensators, reduced order compensators, or compensators with repeated gains, all in a common format. The available performance criteria include H^2, combined H^2/H^∞, maximum entropy, and Popov. The toolbox has been tested on SUN, DEC, HP, SGI, and IBM UNIX workstations, and UNIX *make* files are provided for installation on all of these systems. Documentation for the toolbox is appended to this report. Both the toolbox and documentation are available at the URL:

`http://www.cs.vt.edu/ ltw/toolbox/`

Personnel supported.

Computer Science M.S. student Kelly O'Brien, Ph.D. student Yuan Wang, and the PI Layne Watson were supported by the grant. Faculty and students associated with the grant include Dennis Bernstein (Michigan), Scott Erwin (Michigan), Yuzhen Ge (Butler), and Emmanuel Collins (Florida A&M).

Publications.

Journal articles published and submitted during the grant period are:

Y. Ge, E. G. Collins, Jr., and L. T. Watson, "A comparison of homotopies for alternative formulations of the L^2 optimal model order reduction problem", *J. Comput. Appl. Math.*, 69 (1996) 215–241.

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J. F. Rodríguez, J. E. Renaud, and L. T. Watson, "Trust region augmented Lagrangian methods for sequential response surface approximation and optimization", *ASME J. Mech. Design*, submitted.

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V. Balabanov, M. Kaufman, A. A. Giunta, R. T. Haftka, B. Grossman, W. H. Mason, and L. T. Watson, "Developing customized wing weight function by structural optimization on parallel computers", in *Proc. AIAA/ASME/ASCE/AHS/ASC 37th Structures, Structural Dynamics, and Materials Conf.*, Salt Lake City, UT, AIAA Paper 96–1336, 1996, 113–125.

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Books published during the grant period:

M. Heath, V. Torczon, G. Astfalk, P. E. Bjørstad, A. H. Karp, C. H. Koebel, V. Kumar, R. F. Lucas, L. T. Watson, and D. E. Womble (eds.), *Proceedings of the Eighth SIAM Conference on Parallel Processing for Scientific Computing*, SIAM, Philadelphia, PA, 1997, CD-ROM.

Interactions/transitions.

Conference presentations were:

AIAA/ASME/ASCE/AHS/ASC 37th Structures, Structural Dynamics, and Materials Conference, Salt Lake City, UT, April, 1996.

Copper Mountain Conference on Iterative Methods, Copper Mountain, CO, April, 1996.

First International Conference on Nonlinear Problems in Aviation and Aerospace, Daytona Beach, FL, May, 1996.

Fifth SIAM Conference on Optimization, Victoria, British Columbia, May, 1996 (3 papers).

15th International Conference on Numerical Methods in Fluid Dynamics, Monterey, CA, June, 1996.

13th World Congress of International Federation of Automatic Control, San Francisco, CA, July, 1996.

Approximation Workshop, ICASE, NASA Langley Research Center, Hampton, VA, August, 1996.

Sixth AIAA/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, Bellevue, WA, Sept., 1996 (4 papers).

Society of Engineering Science 33rd Annual Technical Meeting, Tempe, AZ, October, 1996.

INFORMS, Atlanta, GA, Nov., 1996.

35th IEEE Conference on Decision and Control, Kobe, Japan, December, 1996.

8th SIAM Conference on Parallel Processing for Scientific Computing, Minneapolis, MN, March, 1997.

Second World Congress on Structural and Multidisciplinary Optimization, Zakopane, Poland, May, 1997.

1997 American Control Conference, Albuquerque, NM, June, 1997.

Sixth IEEE International Symposium on High Performance Distributed Computing, Portland, OR, August, 1997.

ASME Design Automation Conference, Sacramento, CA, Sept., 1997.

Technology transitions or transfer:

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RESULT
Homotopy algorithms; mathematical software
APPLICATION
Linkage mechanism design; combustion chemistry; robotics; CAD/CAM

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RESULT
Mathematical software
APPLICATION
Bifurcation analysis of control systems

Inventions or patents.

None.

Honors/awards.

IEEE Fellow: Layne T. Watson.

Convergence Theory of Probability-one Homotopies for Model Order Reduction*

Y. WANG†, D. S. BERNSTEIN‡, and L. T. WATSON+

Theory for the global convergence of some probability-one homotopies for the H^2 optimal model order reduction problem is developed.

Key Words—Embedding; globally convergent homotopy; H^2 optimal model order reduction; probability-one homotopy.

Abstract—The optimal H^2 model reduction problem is an inherently nonconvex problem and thus provides a nontrivial computational challenge. This paper systematically examines the requirements of probability-one homotopy methods to guarantee global convergence. Homotopy algorithms for nonlinear systems of equations construct a continuous family of systems and solve the given system by tracking the continuous curve of solutions to the family. The main emphasis is on guaranteeing transversality for several homotopy maps based upon the pseudogramian formulation of the optimal projection equations and variations based upon canonical forms. These results are essential to the probability-one homotopy approach by guaranteeing good numerical properties in the computational implementation of the homotopy algorithms.

1. INTRODUCTION

Numerous techniques have been developed to address the model order reduction problem with both H^2 and H^∞ criteria. Model reduction from an H^2 perspective is considered in (Hyland and Bernstein, 1985) and (Baratchart *et al.*, 1991). Balanced truncation and associated Hankel norm reduction theory are widely used in practice to provide H^∞-suboptimal solutions ((Moore, 1981), (Glover, 1984), (Zhou, 1995), (Kabamba, 1985b)). More recently, convex optimization methods have been employed iteratively to approximate solutions to the nonconvex problem (Grigoriadis, 1995). These techniques are inherently attractive since they rely only upon convexity-based procedures. A more direct albeit computationally challenging approach is to apply fixed-structure optimization (Hyland and Bernstein, 1985), (Haddad and Bernstein, 1989). Special purpose computational methods based upon homotopy techniques have been developed for this problem in (Žigić *et al.*, 1993b), (Ge *et al.*, 1994). The essential difficulties of the model reduction problem are of significant interest since techniques developed for model reduction find immediate application to the closely related problem of reduced-order controller synthesis ((Hyland and Bernstein, 1984), (Haddad and Bernstein, 1990)).

The present paper is concerned with the application of homotopy methods to optimal H^2 model reduction. In computational practice, homotopy methods are widely used for nonconvex optimization ((Watson, 1990), (Watson and Haftka, 1989)). Homotopy methods, in particular, probability-one homotopy methods, have global convergence properties that are often advantageous in comparison to locally convergent methods such as quasi-Newton methods ((Chow *et al.*, 1978), (Watson, 1989), (Watson, 1986)). Under suitable hypotheses, probability-one homotopy methods are guaranteed to converge globally (from an arbitrary starting point) to a solution of a nonlinear system of equations. The nomenclature "probability-one" is well established in the mathematical literature and reflects the generic, measure theoretic properties of the algorithms rather than stochastic aspects.

The goal of the present paper is to systematically examine the requirements of probability-one homotopy methods to guarantee global convergence. The crucial requirements are 1) transversality and 2) boundedness. As discussed in Section 2, transversality implies the existence of and the ability to track a zero curve of the homotopy map, while boundedness is equivalent to the existence of solutions to the model reduction problem. The existence of optimal reduced-order H^2 models follows from the results in (Spanos *et al.*, 1990). The main emphasis in the present paper is on guaranteeing transversality for several homotopy maps based

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upon the pseudogramian formulation of the optimal projection equations and specialized formulations based upon canonical forms. These results are essential to the probability-one homotopy approach by guaranteeing good numerical properties (explained in (Watson *et al.*, 1987)) in the computational implementation of the homotopy algorithms. Numerical comparisons with other approaches have been done elsewhere ((Ge *et al.*, 1996), (Žigić *et al.*, 1993b)), and are not the objective of the present paper.

The contents of the paper are as follows. After stating the H^2 model reduction problem in Section 2, we then provide a brief review of probability-one homotopy theory in Section 3. The transversality assumption of probability-one homotopy theory is then proven in Section 4 for several canonical forms. Next, it is shown by example in Section 5 that the boundedness assumption required by probability-one homotopy theory is not always satisfied by the pseudogramian formulation of the optimal projection equations and by some formulations based upon canonical forms. Then it is shown that for a reformulation of the pseudogramian optimal projection equations in complex projective space using homogeneous transformations, the boundedness assumption holds and thus convergence of the homotopy algorithm to a solution (in complex projective space) is guaranteed. The numerical results in (Ge *et al.*, 1996) and (Žigić *et al.*, 1993b) show that, in practice, it is not necessary to track the homotopy zero curves in complex projective space. Section 6 concludes.

2. H^2 OPTIMAL MODEL ORDER REDUCTION

The H^2 optimal model order reduction problem can be formulated as follows: given the nth-order asymptotically stable, controllable and observable linear time-invariant continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

$$y(t) = Cx(t), \tag{2.2}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and $C \in \mathbf{R}^{l\times n}$, and given $n_m < n$, find an n_mth-order reduced-order model

$$\dot{x}_m(t) = A_m x_m(t) + B_m u(t), \tag{2.3}$$

$$y_m(t) = C_m x_m(t), \tag{2.4}$$

where $A_m \in \mathbf{R}^{n_m\times n_m}$ is asymptotically stable, $B_m \in \mathbf{R}^{n_m\times m}$, $C_m \in \mathbf{R}^{l\times n_m}$, which minimizes the quadratic model-reduction criterion

$$J(A_m, B_m, C_m) \equiv \lim_{t\to\infty} E[(y(t) - y_m(t))^T R(y(t) - y_m(t))] \tag{2.5}$$

where the input $u(t)$ is white noise with positive definite intensity V, and R is a positive definite weighting matrix. Throughout, all positive semidefinite and positive definite matrices are assumed to be symmetric.

To guarantee that J is finite, a solution $(A_m,\ B_m,\ C_m)$ is sought in the set $\mathcal{S} = \{(A_m, B_m, C_m) : A_m$ is asymptotically stable, $(A_m,\ B_m)$ is controllable and $(A_m,\ C_m)$ is observable$\}$. In this case the quadratic model reduction criterion (2.5) is given by

$$J(A_m, B_m, C_m) = \operatorname{tr}[\widetilde{Q}\widetilde{R}], \tag{2.6}$$

where

$$\widetilde{A} \equiv \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}, \qquad \widetilde{B} \equiv \begin{pmatrix} B \\ B_m \end{pmatrix},$$

$$\widetilde{C} \equiv (C \quad -C_m), \qquad \widetilde{R} \equiv \widetilde{C}^T R \widetilde{C}$$

and

$$\widetilde{Q} = \int_0^\infty e^{\widetilde{A}t} \widetilde{B} V \widetilde{B}^T e^{\widetilde{A}^T t}\, dt,$$

which is the unique solution of the Lyapunov equation

$$\widetilde{A}\widetilde{Q} + \widetilde{Q}\widetilde{A}^T + \widetilde{B}V\widetilde{B}^T = 0. \tag{2.7a}$$

For future reference define $\widetilde{P}$ by

$$\widetilde{A}^T\widetilde{P} + \widetilde{P}\widetilde{A} + \widetilde{C}^T R\widetilde{C} = 0, \tag{2.7b}$$

and partition $\widetilde{P}$, $\widetilde{Q}$ as

$$\widetilde{P} = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{pmatrix}, \qquad \widetilde{Q} = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}$$

in conformance with $\widetilde{A}$.

The following theorems and lemmas from (Haddad and Bernstein, 1989), (Hyland and Bernstein, 1985) will be needed in Section 4.

Lemma 2.1. Let positive semidefinite $\hat{Q}, \hat{P} \in \mathbf{R}^{n\times n}$ satisfy

$$\operatorname{rank}(\hat{Q}) = \operatorname{rank}(\hat{P}) = \operatorname{rank}(\hat{Q}\hat{P}) = n_m, \tag{2.8}$$

where $n_m \le n$. Then there exist nonsingular $W \in \mathbf{R}^{n\times n}$ and positive definite diagonal $\Sigma \in \mathbf{R}^{n_m\times n_m}$ such that

$$\hat{Q} = W\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} W^T,\ \hat{P} = W^{-T}\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} W^{-1}.$$

Lemma 2.2. Let positive semidefinite $\hat{Q}, \hat{P} \in \mathbf{R}^{n\times n}$ satisfy the rank conditions (2.8), where $n_m < n$. Then, there exist $G, \Gamma \in \mathbf{R}^{n_m\times n}$ and positive semisimple $M \in \mathbf{R}^{n_m\times n_m}$, unique up to a change of basis in $\mathbf{R}^{n_m}$, such that

$$\hat{Q}\hat{P} = G^T M \Gamma, \quad \Gamma G^T = I_{n_m}. \tag{2.9}$$

Theorem 2.3. Suppose $(A_m,\ B_m,\ C_m) \in \mathcal{S}$ solves the optimal model-reduction problem. Then there exist positive semidefinite matrices $\hat{Q}$,

$\hat{P} \in \mathbf{R}^{n\times n}$ satisfying (2.8) and such that A_m, B_m and C_m are given by

$$A_m = \Gamma A G^T,\ B_m = \Gamma B,\ C_m = C G^T, \quad (2.10)$$

and such that, with $\tau \equiv G^T\Gamma$, the following conditions are satisfied:

$$\tau[A\,\hat{Q} + \hat{Q}\,A^T + B\,V\,B^T] = 0, \quad (2.11)$$

$$[A^T\,\hat{P} + \hat{P}\,A + C^T\,R\,C]\,\tau = 0. \quad (2.12)$$

3. PROBABILITY-ONE GLOBALLY CONVERGENT HOMOTOPIES

A *homotopy* is a continuous map from the interval [0,1] into a function space, where the continuity is with respect to the topology of the function space. Intuitively, a homotopy $\rho(\lambda)$ continuously deforms the function $\rho(0) = g$ into the function $\rho(1) = f$ as λ goes from 0 to 1. In this case, f and g are said to be ***homotopic***. Homotopy maps are fundamental tools in topology, and provide a powerful mechanism for defining equivalence classes of functions.

Homotopies provide a mathematical formalism for describing an old procedure in numerical analysis, variously known as continuation, incremental loading, and embedding. The continuation procedure for solving a nonlinear system of equations $f(x) = 0$ starts with a (generally simpler) problem $g(x) = 0$ whose solution x_0 is known. The continuation procedure is to track the set of zeros of

$$\rho(\lambda, x) = \lambda f(x) + (1-\lambda)g(x) \quad (3.1)$$

as λ is increased monotonically from 0 to 1, starting at the known initial point $(0, x_0)$ satisfying $\rho(0, x_0) = 0$. Each step of this tracking process is done by starting at a point $(\tilde{\lambda}, \tilde{x})$ on the zero set of ρ, fixing some $\Delta\lambda > 0$, and then solving $\rho(\tilde{\lambda} + \Delta\lambda, x) = 0$ for x using a locally convergent iterative procedure, which requires an invertible Jacobian matrix $D_x\rho(\tilde{\lambda} + \Delta\lambda, x)$. The process stops at $\lambda = 1$, since $f(\bar{x}) = \rho(1, \bar{x}) = 0$ gives a zero $\bar{x}$ of $f(x)$. Note that continuation assumes that the zeros of ρ connect the zero x_0 of g to a zero $\bar{x}$ of f, and that the Jacobian matrix $D_x\rho(\lambda, x)$ is invertible along the zero set of ρ; these are strong assumptions, which are frequently not satisfied in practice.

Continuation can fail because the curve γ of zeros of $\rho(\lambda, x)$ emanating from $(0, x_0)$ may (1) have turning points, (2) bifurcate, (3) fail to exist at some λ values, or (4) wander off to infinity without reaching $\lambda = 1$. Turning points and bifurcation correspond to singular $D_x\rho(\lambda, x)$. Generalizations of continuation known as *homotopy methods* attempt to deal with cases (1) and (2), and allow tracking of γ to continue through singularities. In particular, continuation monotonically increases λ, whereas homotopy methods permit λ to both increase and decrease along γ. Homotopy methods can also fail via cases (3) or (4).

The map $\rho(\lambda, x)$ connects the functions $g(x)$ and $f(x)$, hence the use of the word "homotopy." In general the homotopy map $\rho(\lambda, x)$ need not be a simple convex combination of g and f as in (3.1), and can involve λ nonlinearly. Sometimes λ is a physical parameter in the original problem $f(x; \lambda) = 0$, where $\lambda = 1$ is the (nondimensionalized) value of interest, although "artificial parameter" homotopies are generally more computationally efficient than "natural parameter" homotopies $\rho(\lambda, x) = f(x; \lambda)$. An example of an artificial parameter homotopy map is

$$\rho(\lambda, x) = \lambda f(x; \lambda) + (1-\lambda)(x-a), \quad (3.2)$$

which satisfies $\rho(0, a) = 0$. The name "artificial" reflects the fact that solutions to $\rho(\lambda, x) = 0$ have no physical interpretation for $\lambda < 1$. Note that $\rho(\lambda, x)$ in (3.2) has a unique zero $x = a$ at $\lambda = 0$, regardless of the structure of $f(x; \lambda)$.

All four shortcomings of continuation and homotopy methods have been overcome by probability-one homotopies, proposed in 1976 by Chow, Mallet-Paret, and Yorke (Chow *et al.*, 1978). The supporting theory, based on differential geometry, will be reformulated in less technical jargon here.

Definition 3.1. Let $U \subset \mathbf{R}^m$ and $V \subset \mathbf{R}^p$ be open sets, and let $\rho: U \times [0,1) \times V \to \mathbf{R}^p$ be a C^2 map. ρ is said to be *transversal to zero* if the $p \times (m+1+p)$ Jacobian matrix $D\rho$ has full rank on $\rho^{-1}(0)$.

The C^2 requirement is technical, and part of the definition of transversality. The basis for the probability-one homotopy theory is:

Theorem 3.2 (Parametrized Sard's Theorem) (Chow *et al.*, 1978). Let $\rho: U \times [0,1) \times V \to \mathbf{R}^p$ be a C^2 map. If ρ is transversal to zero, then for almost all $a \in U$ the map

$$\rho_a(\lambda, x) = \rho(a, \lambda, x)$$

is also transversal to zero.

To discuss the import of this theorem, take $U = \mathbf{R}^m, V = \mathbf{R}^p$, and suppose that the C^2 map $\rho: \mathbf{R}^m \times [0,1) \times \mathbf{R}^p \to \mathbf{R}^p$ is transversal to zero. A straightforward application of the implicit function theorem yields that for almost all $a \in \mathbf{R}^m$, the zero set of ρ_a consists of smooth, nonintersecting curves which either (1) are closed loops lying entirely in $(0,1) \times \mathbf{R}^p$, (2) have both endpoints in $\{0\} \times \mathbf{R}^p$, (3) have both endpoints in $\{1\} \times \mathbf{R}^p$, (4) are unbounded with one endpoint in either $\{0\} \times \mathbf{R}^p$ or in $\{1\} \times \mathbf{R}^p$, or (5) have one endpoint in $\{0\} \times \mathbf{R}^p$

and the other in $\{1\} \times \mathbf{R}^p$. Furthermore, for almost all $a \in \mathbf{R}^m$, the Jacobian matrix $D\rho_a$ has full rank at *every* point in $\rho_a^{-1}(0)$. The goal is to construct a map ρ_a whose zero set has an endpoint in $\{0\} \times \mathbf{R}^p$, and which rules out (2) and (4). Then (5) obtains, and a zero curve starting at $(0, x_0)$ is *guaranteed* to reach a point $(1, \bar{x})$. All of this holds for almost all $a \in \mathbf{R}^m$, and hence with probability one (Chow *et al.*, 1978). Furthermore, since $a \in \mathbf{R}^m$ can be almost any point (and, indirectly, so can the starting point x_0), an algorithm based on tracking the zero curve in (5) is legitimately called *globally convergent*. This discussion is summarized in the following theorem.

Theorem 3.3. Let $f : \mathbf{R}^p \to \mathbf{R}^p$ be a C^2 map, $\rho : \mathbf{R}^m \times [0,1) \times \mathbf{R}^p \to \mathbf{R}^p$ a C^2 map, and $\rho_a(\lambda, x) = \rho(a, \lambda, x)$. Suppose that

(1) ρ is transversal to zero,

and, for each fixed $a \in \mathbf{R}^m$,

(2) $\rho_a(0, x) = 0$ has a unique solution x_0,

(3) $\rho_a(1, x) = f(x) \qquad (x \in \mathbf{R}^p)$.

Then, for almost all $a \in \mathbf{R}^m$, there exists a zero curve γ of ρ_a emanating from $(0, x_0)$, along which the Jacobian matrix $D\rho_a$ has full rank. If, in addition,

(4) $\rho_a^{-1}(0)$ is bounded,

then γ reaches a point $(1, \bar{x})$ such that $f(\bar{x}) = 0$. Furthermore, if $Df(\bar{x})$ is invertible, then γ has finite arc length.

Any algorithm for tracking γ from $(0, x_0)$ to $(1, \bar{x})$, based on a homotopy map satisfying the hypotheses of Theorem 3.3, is called a *globally convergent probability-one homotopy algorithm*. Of course the practical numerical details of tracking γ are nontrivial, and have been the subject of twenty years of research in numerical analysis. Production quality software called HOMPACK (Watson *et al.*, 1987) exists for tracking γ. The distinctions between continuation, homotopy methods, and probability-one homotopy methods are subtle but worth noting. Only the latter are provably globally convergent and (by construction) expressly avoid dealing with singularities numerically, unlike continuation and homotopy methods which must explicitly handle singularities numerically.

The purpose of this paper is to prove or disprove properties (1)–(4) of Theorem 3.3 for some homotopy maps that have been proposed for the H^2 optimal model order reduction problem, and which have been successful in practice. Assumptions (2) and (3) in Theorem 3.3 are usually achieved by the construction of ρ (such as (3.2)), and are straightforward to verify. Although assumption (1) is trivial to verify for some maps, for the H^2 model order reduction homotopies the verification is nontrivial. Assumption (4) is typically very hard to verify, and often is a deep result, since (1)–(4) holding implies the *existence* of a solution to $f(x) = 0$.

Note that (1)–(4) are sufficient, but not necessary, for the existence of a solution to $f(x) = 0$, which is why homotopy maps not satisfying the hypotheses of Theorem 3.3 can still be very successful on practical problems. If (1)–(3) hold and a solution does *not* exist, then (4) must fail, and nonexistence is manifested by γ going off to infinity. Properties (1)–(3) are important because they guarantee good numerical properties along the zero curve γ, which, if bounded, results in a *globally convergent* algorithm. If γ is unbounded, then either the homotopy approach (with this particular ρ) has failed or $f(x) = 0$ has no solution.

4. TRANSVERSALITY OF HOMOTOPIES FOR H^2 OPTIMAL MODEL ORDER REDUCTION

This section proves that three homotopies $\rho(a, \lambda, x)$ which have been used in (Žigić *et al.*, 1993b) and (Ge *et al.*, 1994) for the H^2 optimal model order reduction problem are transversal to zero, the first requirement of Theorem 3.3. An overview and comparison of these homotopy maps is in (Ge *et al.*, 1996). The analysis concerns (2.11) and (2.12) where $\hat{Q}$ and $\hat{P}$ are positive semidefinite matrices satisfying (2.8).

4.1. Transversality of homotopies based on decompositions of pseudogramians

Since $\hat{Q}$ and $\hat{P}$ satisfy (2.8), there exists invertible $W \in \mathbf{R}^{n \times n}$ and positive definite diagonal $\Sigma \in \mathbf{R}^{n_m \times n_m}$ such that (Hyland and Bernstein, 1985)

$$\hat{Q} = W \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} W^T = W_1 \Sigma W_1^T,$$

$$\hat{P} = W^{-T} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} W^{-1} = U_1^T \Sigma U_1$$

where

$$W = (\overbrace{W_1}^{n_m} \quad W_2), \quad W^{-1} = U = \begin{matrix} n_m\{ \\ \end{matrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}.$$

Premultipling (2.11) by U_1 and postmultiplying (2.12) by W_1 yields (recall that $\tau = G^T \Gamma = W_1 U_1$)

$$U_1 A W_1 \Sigma W_1^T + \Sigma W_1^T A^T + U_1 B V B^T = 0, \quad (4.1)$$

$$A^T U_1^T \Sigma + U_1^T \Sigma U_1 A W_1 + C^T R C W_1 = 0. \quad (4.2)$$

A constraint from $W^{-1} = U$ is

$$U_1 W_1 - I = 0. \qquad (4.3)$$

The matrix equations (4.1)–(4.3) contain $2\,n\,n_m + n_m^2$ scalar equations. However, the only unknowns in (4.1)–(4.3), namely W_1, U_1, and diagonal Σ, contain $2\,n\,n_m + n_m$ variables. Hence, some other formulation is necessary in order to make an exact match between the number of equations and the number of unknowns. Following (Žigić *et al.*, 1993b), all n_m^2 elements of Σ are considered as unknowns, giving the same number of equations as unknowns. The structure of the problem is such that Σ will turn out to be symmetric, so it can be diagonalized to produce the decomposition of $\hat{Q}$ and $\hat{P}$ described above.

The approach in (Žigić *et al.*, 1993b), analyzed next, uses the homotopy map

$$\rho_a(\lambda, x) \equiv \lambda\, f(x) + (1-\lambda) g(x; a), \qquad (4.4)$$

where the initial problem $\rho_a(0,x) = g(x;a) = 0$ has an easily obtained unique solution and the final problem (4.1)–(4.3) is $\rho_a(1,x) = f(x) = 0$. f and g are displayed in (4.4) simply to point out that the map $\rho_a(\lambda,x)$ can be viewed as a convex combination of two other maps. For notational convenience later when displaying Jacobian matrices the order of the variables is henceforth taken as λ, x, a. Let

$$A(\lambda) = A, \quad B(\lambda) = \lambda B V B^T + (1-\lambda) B_i,$$
$$C(\lambda) = \lambda C^T R C + (1-\lambda) C_i,$$

where $B_i = B(0)$ and $C_i = C(0)$ are matrices defining the initial problem at $\lambda = 0$, and correspond to the parameter vector a in Theorem 3.3. Define

$$\rho_a(\lambda, x) \equiv \rho(\lambda, x, a) \equiv \begin{pmatrix} F_1(\lambda,x,a) \\ F_2(\lambda,x,a) \\ F_3(\lambda,x,a) \end{pmatrix}$$

in (4.4) by

$$F_1(\lambda,x,a) \equiv U_1\, A(\lambda)\, W_1\, \Sigma\, W_1^T + \Sigma\, W_1^T\, A^T(\lambda) + U_1\, B(\lambda), \qquad (4.5)$$

$$F_2(\lambda,x,a) \equiv A^T(\lambda)\, U_1^T\, \Sigma + U_1^T\, \Sigma\, U_1\, A(\lambda)\, W_1 + C(\lambda)\, W_1, \qquad (4.6)$$

$$F_3(\lambda,x,a) \equiv U_1\, W_1 - I, \qquad (4.7)$$

where

$$a \equiv \begin{pmatrix} \mathrm{Vec}\,(B_i) \\ \mathrm{Vec}\,(C_i) \end{pmatrix}$$

is the generic parameter vector in Theorem 3.3 and in (4.4),

$$x \equiv \begin{pmatrix} \mathrm{Vec}\,(W_1) \\ \mathrm{Vec}\,(U_1) \\ \mathrm{Vec}\,(\Sigma) \end{pmatrix}$$

denotes the independent variables $W_1 \in \mathbf{R}^{n \times n_m}$, $U_1 \in \mathbf{R}^{n_m \times n}$, $\Sigma \in \mathbf{R}^{n_m \times n_m}$ corresponding to x in Theorem 3.3, and A, B, C, V, R are constants as in Section 2.

The Jacobian matrix of $\rho(\lambda,x,a)$ has $2\,n\,n_m + n_m^2$ rows and $2\,n^2 + 2\,n\,n_m + n_m^2 + 1$ columns. Rows 1 through $n\,n_m$ correspond to (4.5), rows $n\,n_m + 1$ through $2\,n\,n_m$ correspond to (4.6), and rows $2\,n\,n_m + 1$ through $2\,n\,n_m + n_m^2$ correspond to (4.7). The first column corresponds to the derivatives with respect to λ, columns 2 through $n\,n_m + 1$ correspond to the derivatives with respect to W_1, columns $n\,n_m + 2$ through $2\,n\,n_m + 1$ correspond to the derivatives with respect to U_1, columns $2\,n\,n_m + 2$ through $2\,n\,n_m + n_m^2 + 1$ correspond to the derivatives with respect to Σ, columns $2\,n\,n_m + n_m^2 + 2$ through $2\,n\,n_m + n_m^2 + n^2 + 1$ correspond to the derivatives with respect to B_i, and columns $2\,n\,n_m + n_m^2 + n^2 + 2$ through $2\,n\,n_m + n_m^2 + 2\,n^2 + 1$ correspond to the derivatives with respect to C_i:

$$D\rho(\lambda,x,a) = \begin{pmatrix} D_\lambda \rho & D_{W_1}\rho & D_{U_1}\rho & D_\Sigma \rho & D_{B_i}\rho & D_{C_i}\rho \end{pmatrix}. \qquad (4.8)$$

Since $F_3(\lambda,x,a)$ does not depend upon λ, B_i, and C_i, it follows that

$$D_\lambda F_3(\lambda,x,a) = 0,$$
$$D_{B_i} F_3(\lambda,x,a) = 0,$$
$$D_{C_i} F_3(\lambda,x,a) = 0,$$

and similarly

$$D_{C_i} F_1(\lambda,x,a) = D_{B_i} F_2(\lambda,x,a) = 0.$$

Thus

$$D\rho(\lambda,x,a) = D\rho(\lambda, W_1, U_1, \Sigma, B_i, C_i) = \begin{pmatrix} D_\lambda F_1 & D_x F_1 & D_a F_1 \\ D_\lambda F_2 & D_x F_2 & D_a F_2 \\ 0 & D_x F_3 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} D_\lambda F_1 & D_{W_1} F_1 & D_{U_1} F_1 & D_\Sigma F_1 & D_{B_i} F_1 & 0 \\ D_\lambda F_2 & D_{W_1} F_2 & D_{U_1} F_2 & D_\Sigma F_2 & 0 & D_{C_i} F_2 \\ 0 & D_{W_1} F_3 & D_{U_1} F_3 & D_\Sigma F_3 & 0 & 0 \end{pmatrix}. \qquad (4.9)$$

The following lemma will be used in the proof of Theorem 4.2.

Lemma 4.1. Let $X \in \mathbf{R}^{p \times q}$ and $A \in \mathbf{R}^{n \times m}$, $B \in \mathbf{R}^{m \times l}$ be differentiable with respect to x_{ij} for $1 \le i \le p$, $1 \le j \le q$. Then

$$\frac{\partial}{\partial x_{ij}}(A\,B) = (\frac{\partial}{\partial x_{ij}} A) B + A(\frac{\partial}{\partial x_{ij}} B),$$

and for constant M, interpreting the derivative D_X as $D_{\mathrm{Vec}(X)}$,

$$D_X(MX) = I \otimes M, \qquad D_X(XM) = M^T \otimes I.$$

The proof of Lemma 4.1 is straightforward calculus.

Theorem 4.2. The homotopy map given by (4.5)–(4.7) is transversal to zero (for $0 \le \lambda < 1$).

Proof. To prove that $D\rho(\lambda, x, a)$ given in (4.9) has full rank, i.e.,

$$\text{rank } (D\rho(\lambda, x, a)) = 2nn_m + n_m^2,$$

it suffices to prove that

$$\begin{aligned}\text{rank}(D_x F_3) &= \text{rank}(D_{W_1} F_3 \quad D_{U_1} F_3 \quad D_\Sigma F_3) \\ &= n_m^2, \qquad (4.10)\end{aligned}$$

$$\text{rank}(D_a F_1) = \text{rank}(D_{B_i} F_1 \quad 0) = nn_m, \quad (4.11)$$

$$\text{rank}(D_a F_2) = \text{rank}(0 \quad D_{C_i} F_2) = nn_m. \quad (4.12)$$

The meaning of expressions like $D_\Sigma F_3$ is ambiguous until some ordering is specified for the components of the matrices Σ and F_3. Hereafter, whichever ordering is notationally convenient is used. If unspecified, the standard ordering by columns (Vec) is assumed.

Using Lemma 4.1, ordering U_1 and F_3 by rows,

$$D_{U_1} F_3(\lambda, x, a) = D_{U_1}(U_1 W_1) = I_{n_m} \otimes W_1^T, \quad (4.13)$$

and ordering W_1 and F_3 by columns,

$$D_{W_1} F_3(\lambda, x, a) = D_{W_1}(U_1 W_1) = I_{n_m} \otimes U_1. \quad (4.14)$$

Since $U_1 W_1 = I$, by Sylvester's inequality,

$$\text{rank } (U_1) = \text{rank } (W_1) = n_m,$$

and therefore

$$\begin{aligned}\text{rank}(D_x F_3) &= \text{rank}(D_{U_1} F_3) \\ &= \text{rank}(D_{W_1} F_3) = n_m^2,\end{aligned}$$

which is (4.10).

Using Lemma 4.1, ordering B_i and F_1 by columns yields

$$\begin{aligned}D_{B_i} F_1(\lambda, x, a) &= D_{B_i}(U_1 B(\lambda)) \\ &= (1-\lambda) D_{B_i}(U_1 B_i) \\ &= (1-\lambda) I_n \otimes U_1, \qquad (4.15)\end{aligned}$$

and using (4.15) for $\lambda < 1$ yields

$$\text{rank}(D_{B_i} F_1) = nn_m.$$

Similarly, ordering C_i and F_2 by rows,

$$\begin{aligned}D_{C_i} F_2(\lambda, x, a) &= D_{C_i}(C(\lambda) W_1) \\ &= (1-\lambda) D_{C_i}(C_i W_1) \\ &= (1-\lambda) I_n \otimes W_1^T, \quad (4.16)\end{aligned}$$

so for $\lambda < 1$

$$\text{rank}(D_{C_i} F_2) = nn_m.$$

This completes the proof of (4.10)–(4.12), and the proof that the homotopy map (4.5)–(4.7) is transversal to zero for all $0 \le \lambda < 1$. Q. E. D.

Remark 4.2.1. One can use more variables in the parameter vector a, e.g., $A(\lambda) = \lambda A + (1-\lambda) A_i$, without affecting the full rank properties.

4.2. *Transversality of homotopies based on input normal form*

The following theorem from (Kabamba, 1985a) is needed to present the homotopy method for the input normal form.

Theorem 4.3. Suppose $(\bar{A}_m, \bar{B}_m, \bar{C}_m)$ is asymptotically stable and minimal. Then there exist a similarity transformation U and a positive definite matrix $\Omega = \text{diag}(\omega_1, \cdots, \omega_{n_m})$ such that $A_m = U^{-1}\bar{A}_m U$, $B_m = U^{-1}\bar{B}_m$, and $C_m = \bar{C}_m U$ satisfy

$$\begin{aligned}A_m + A_m^T + B_m V B_m^T &= 0, \\ A_m^T \Omega + \Omega A_m + C_m^T R C_m &= 0.\end{aligned} \qquad (4.17)$$

In addition, if the ω_i are distinct,

$$\begin{aligned}(A_m)_{ii} &= -\frac{1}{2}(B_m V B_m^T)_{ii}, \\ \omega_i &= \frac{(C_m^T R C_m)_{ii}}{(B_m V B_m^T)_{ii}}, \\ (A_m)_{ij} &= \frac{(C_m^T R C_m)_{ij} - \omega_j (B_m V B_m^T)_{ij}}{\omega_j - \omega_i}.\end{aligned} \qquad (4.18)$$

Definition 4.3.1. The triple (A_m, B_m, C_m) satisfying (4.17) and (4.18) is said to be in *input normal form.*

The utility of the input normal form (4.17)-(4.18) lies in using B_m and C_m as the independent variables, and then being able to recover A_m uniquely from B_m and C_m. The number of variables in B_m and C_m is $n_m(m+l)$, the minimum number of variables possible to describe any reduced order model, and thus the input normal form parametrization is referred to as a "minimal parametrization." If $\omega_i = \omega_j$ for some $i \ne j$, then, regardless of (4.17) holding, (4.18) fails to permit the unique recovery of A_m.

Under the assumption that the solution (A_m, B_m, C_m) being sought exists in input normal form, the only independent variables are B_m and C_m, and in this case the domain is

$\{(A_m, B_m, C_m) :\ A_m$ is asymptotically stable, (A_m, B_m, C_m) is minimal and in input normal form$\}$.

Now for (A_m, B_m, C_m) in input normal form, the cost function can be written as

$$J(A_m, B_m, C_m) = \text{tr } (\tilde{Q}_I \tilde{R}_I), \qquad (4.19)$$

where $\tilde{Q}_I$ is a symmetric and positive definite matrix satisfying

$$\tilde{A}_I \tilde{Q}_I + \tilde{Q}_I \tilde{A}_I^T + \tilde{V}_I = 0, \qquad (4.20)$$

and

$$\tilde{A}_I = \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix},$$

$$\tilde{R}_I = \begin{pmatrix} C^T RC & -C^T RC_m \\ -C_m^T RC & C_m^T RC_m \end{pmatrix},$$

$$\tilde{V}_I = \begin{pmatrix} BVB^T & BVB_m^T \\ B_m VB^T & B_m VB_m^T \end{pmatrix}. \quad (4.21)$$

$\tilde{Q}_I$ can be written as

$$\tilde{Q}_I = \begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^T & \bar{Q}_2 \end{pmatrix}, \quad (4.22)$$

where $\bar{Q}_1 \in \mathbf{R}^{n\times n}$, $\bar{Q}_{12} \in \mathbf{R}^{n\times n_m}$, and $\bar{Q}_2 \in \mathbf{R}^{n_m\times n_m}$.

Minimizing (4.19) under the constraints (4.17) and (4.20) leads to the Lagrangian

$$\begin{aligned} &L(A_m, B_m, C_m, \Omega, \tilde{Q}_I, M_c, M_0, \tilde{P}_I) \\ &= \mathrm{tr}\Big[\tilde{Q}_I\tilde{R}_I + (A_m + A_m^T + B_m V B_m^T)M_c \\ &+ (A_m^T\Omega + \Omega A_m + C_m^T RC_m)M_o \\ &+ (\tilde{A}_I\tilde{Q}_I + \tilde{Q}_I\tilde{A}_I^T + \tilde{V}_I)\tilde{P}_I\Big], \end{aligned}$$

where the symmetric matrices M_o, M_c, and $\tilde{P}_I$ are Lagrange multipliers.

Setting $\partial L/\partial\tilde{Q}_I = 0$ gives an equation for $\tilde{P}_I$ similar to (4.20) for $\tilde{P}$,

$$\tilde{A}_I^T\tilde{P}_I + \tilde{P}_I\tilde{A}_I + \tilde{R}_I = 0, \quad (4.23)$$

where $\tilde{P}_I$ is symmetric positive definite and can be partitioned similarly to $\tilde{Q}_I$ as

$$\tilde{P}_I = \begin{pmatrix} \bar{P}_1 & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_2 \end{pmatrix}. \quad (4.24)$$

The matrices M_c and M_o satisfy (Davis *et al.*, 1992)

$$M_c = -(\frac{1}{2}S + \Omega M_o), \quad (4.25)$$

$$(M_o)_{ii} = -\frac{1}{(A_m)_{ii}}\sum_{\substack{j=1\\ j\neq i}}^{n_m}(A_m)_{ij}(M_o)_{ji}, \quad (4.26)$$

$$(M_o)_{ij} = \frac{(S)_{ij} - (S)_{ji}}{2(\omega_j - \omega_i)}, \quad \text{if } \omega_j \neq \omega_i, \quad (4.27)$$

where $S = 2(\bar{P}_{12}^T\bar{Q}_{12} + \bar{P}_2\bar{Q}_2)$.

Setting $\partial L/\partial B_m = 0$ and $\partial L/\partial C_m = 0$ gives

$$2(\bar{P}_{12}^T B + \bar{P}_2 B_m)V + 2M_c B_m V = 0, \quad (4.28)$$

$$2R(C_m\bar{Q}_2 - C\bar{Q}_{12}) + 2RC_m M_o = 0. \quad (4.29)$$

Observe that $\tilde{P}_I$ through (4.23) and $\tilde{Q}_I$ through (4.20) depend on B_m and C_m as does A_m through (4.18). Similarly M_c through (4.25) and M_0 through (4.26)–(4.27) depend on B_m and C_m. Thus everything in (4.28)–(4.29) is a function of B_m and C_m. Use the homotopy map structure of (4.4) and let

$$B(\lambda) = \lambda B + (1-\lambda)B_i,$$
$$C(\lambda) = \lambda C + (1-\lambda)C_i,$$

where B_i and C_i are matrices defining the initial problem at $\lambda = 0$, and correspond to the parameter vector a in Theorem 3.3. The structure of the homotopy map $\rho(\lambda, x, a)$ for the input normal form is now

$$F_1(\lambda, x, a) = (\bar{P}_{12}^T B(\lambda) + \bar{P}_2 B_m)V + M_c B_m V, \quad (4.30)$$

$$F_2(\lambda, x, a) = R(C_m\bar{Q}_2 - C(\lambda)\bar{Q}_{12}) + RC_m M_o, \quad (4.31)$$

where

$$a \equiv \begin{pmatrix} \mathrm{Vec}\ (B_i) \\ \mathrm{Vec}\ (C_i) \end{pmatrix}$$

denotes the parameter variables $B_i \in \mathbf{R}^{n\times m}$, $C_i \in \mathbf{R}^{l\times n}$,

$$x \equiv \begin{pmatrix} \mathrm{Vec}\ (B_m) \\ \mathrm{Vec}\ (C_m) \end{pmatrix}$$

denotes the independent variables B_m and C_m corresponding to x in Theorem 3.3, and A, B, C, V, R are constants as in Section 2.

The Jacobian matrix of $\rho(\lambda, x, a)$ has $n_m m + n_m l$ rows and $(n_m + n)(m + l) + 1$ columns. Since $F_1(\lambda, x, a)$ does not involve C_i and $F_2(\lambda, x, a)$ does not involve B_i

$$D_{C_i}F_1(\lambda, x, a) = 0, \qquad D_{B_i}F_2(\lambda, x, a) = 0.$$

The Jacobian matrix is

$$D\rho(\lambda, x, a) = \begin{pmatrix} D_\lambda F_1 & D_{B_m}F_1 & D_{C_m}F_1 & D_{B_i}F_1 & 0 \\ D_\lambda F_2 & D_{B_m}F_2 & D_{C_m}F_2 & 0 & D_{C_i}F_2 \end{pmatrix}. \quad (4.32)$$

The following lemma will be used in the proof of Theorem 4.5.

Lemma 4.4. Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{A}_I$, $\tilde{B}_I$, $\tilde{C}_I$, $\tilde{P}$, $\tilde{Q}$, $\tilde{R}$, $\tilde{P}_I$, $\tilde{Q}_I$, $\tilde{R}_I$, Ω and U be defined as above. Then

$$\bar{Q}_1 = Q_1, \qquad \bar{P}_1 = P_1, \quad (4.33)$$

$$\bar{Q}_{12} = Q_{12}U^{-T}, \qquad \bar{P}_{12} = P_{12}U, \quad (4.34)$$

$$\bar{Q}_2 = I, \qquad \bar{P}_2 = \Omega, \quad (4.35)$$

$$Q_2 = UU^T, \qquad P_2 = U^{-T}\Omega U^{-1}. \quad (4.36)$$

In addition, P_{12}, Q_{12}, $\bar{P}_{12}$, and $\bar{Q}_{12}$ have full column rank.

Proof. Equations (4.20) and (4.23) can be written in the form

$$\begin{aligned} &\begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}\begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^T & \bar{Q}_2 \end{pmatrix} \\ &+ \begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^T & \bar{Q}_2 \end{pmatrix}\begin{pmatrix} A^T & 0 \\ 0 & A_m^T \end{pmatrix} \\ &+ \begin{pmatrix} BVB^T & BVB_m^T \\ B_m VB^T & B_m VB_m^T \end{pmatrix} = 0, \end{aligned}$$

$$\begin{pmatrix} A^T & 0 \\ 0 & A_m^T \end{pmatrix}\begin{pmatrix} \bar{P}_1 & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_2 \end{pmatrix}$$

$$+\begin{pmatrix} \bar{P}_1 & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_2 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}$$
$$+\begin{pmatrix} C^TRC & -C^TRC_m \\ -C_m^TRC & C_m^TRC_m \end{pmatrix} = 0.$$

Expanding these equations yields

$$A\bar{Q}_1 + \bar{Q}_1A^T + BVB^T = 0, \quad (4.37)$$
$$A\bar{Q}_{12} + \bar{Q}_{12}A_m^T + BVB_m^T = 0, \quad (4.38)$$
$$A_m\bar{Q}_2 + \bar{Q}_2A_m^T + B_mVB_m^T = 0, \quad (4.39)$$
$$A^T\bar{P}_1 + \bar{P}_1A + C^TRC = 0, \quad (4.40)$$
$$A^T\bar{P}_{12} + \bar{P}_{12}A_m - C^TRC_m = 0, \quad (4.41)$$
$$A_m^TP_2 + \bar{P}_2A_m + C_m^TRC_m = 0. \quad (4.42)$$

Comparing (2.7a) with (4.37), and (2.7b) with (4.40) yields (4.33).

If the definitions $A_m = U^{-1}\bar{A}_mU$, $B_m = U^{-1}\bar{B}_m$, and $C_m = \bar{C}_mU$ in Theorem 4.3 are substituted into (4.17) then (4.17) becomes

$$\bar{A}_mUU^T + UU^T\bar{A}_m^T + \bar{B}_mV\bar{B}_m^T = 0, \quad (4.43)$$
$$\bar{A}_m^TU^{-T}\Omega U^{-1} + U^{-T}\Omega U^{-1}\bar{A}_m + \bar{C}_m^TR\bar{C}_m = 0. \quad (4.44)$$

Comparing (2.7a) and (2.7b) with (4.43) and (4.44) yields (4.36).

If $A_m = U^{-1}\bar{A}_mU$, $B_m = U^{-1}\bar{B}_m$, and $C_m = \bar{C}_mU$ are substituted into (4.38) and (4.41) and the resulting equations are compared with (2.7a) and (2.7b), then (4.34) follows. Comparing (4.17) and (4.18) with (4.39) and (4.42) yields (4.35).

Finally, since Q_2 and P_2 are nonsingular, from Section 6 in (Ge *et al.*, 1996) it follows that Q_{12} and P_{12} have full column rank. Since U is nonsingular, from (4.34) it follows that $\bar{Q}_{12}$ and $\bar{P}_{12}$ also have full rank. Q. E. D.

Theorem 4.5. Let $\bar{P}_I$ and $\bar{Q}_I$ be defined as above. Then $D\rho(\lambda, x, a)$ given by (4.32) has full column rank for $0 \le \lambda < 1$, i.e., the homotopy map (4.30)–(4.31) is transversal to zero for $0 \le \lambda < 1$.

Proof. To prove $D\rho(\lambda, x, a)$ given by (4.32) has full column rank, i.e.,

$$\text{rank}\,(D\rho(\lambda, x, a)) = n_m m + n_m l,$$

it suffices to prove that

$$\text{rank}(D_aF_1) = \text{rank}(D_{B_i}F_1) = n_m m, \quad (4.45)$$
$$\text{rank}(D_aF_2) = \text{rank}(D_{C_i}F_2) = n_m l. \quad (4.46)$$

Since V and R are constant symmetric positive definite matrices, without loss of generality set $V = I$ in (4.30) and $R = I$ in (4.31) . Using Lemma 4.1 to compute $D_{B_i}F_1(\lambda, x, a)$, ordering B_i and F_1 by columns,

$$\begin{aligned} D_{B_i}F_1(\lambda, x, a) &= D_{B_i}(\bar{P}_{12}^TB(\lambda)) \\ &= (1-\lambda)D_{B_i}(\bar{P}_{12}^TB_i) \\ &= (1-\lambda)I_m \otimes \bar{P}_{12}^T. \end{aligned} \quad (4.47)$$

Ordering C_i and F_2 by rows gives

$$\begin{aligned} D_{C_i}F_2(\lambda, x, a) &= D_{C_i}(-C(\lambda)\bar{Q}_{12}) \\ &= (\lambda-1)D_{C_i}(C_i\bar{Q}_{12}) \\ &= (\lambda-1)I_l \otimes \bar{Q}_{12}^T. \end{aligned} \quad (4.48)$$

Now finally, using Lemma 4.4, (4.47), and (4.48), the rank statements of (4.45) and (4.46) follow.

Thus the homotopy map (4.30)–(4.31) for the input normal form parametrization of (A_m, B_m, C_m) for the H^2 model order reduction problem is transversal to zero. Q. E. D.

4.3. *Transversality of homotopies based on Ly's formulation*

In Ly's formulation (Ly *et al.*, 1985), the reduced order model is represented with respect to a basis such that A_m is a 2×2 block-diagonal matrix (2×2 blocks with an additional 1×1 block if n_m is odd) with 2×2 blocks in the form

$$\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix},$$

B_m is a full matrix, and $C_m = \big((C_m)_1\ (C_m)_2 \cdots (C_m)_i \cdots (C_m)_r\big)$, where

$$(C_m)_i = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \end{pmatrix}^T,$$
$$(C_m)_r = (1 \quad * \quad \cdots \quad *)^T, \quad \text{if } n_m \text{ is odd.}$$

Let $\mathcal{S}$ be the set of indices of those elements of A_m which are independent variables, i.e., $\mathcal{S} \equiv \{(2,1),\ (2,2),\ \ldots,\ (2i,2i-1),\ (2i,2i),\ \ldots,\ (n_m,n_m)\}$. To minimize the cost function $J(A_m, B_m, C_m)$, consider the Lagrangian

$$L(A_m, B_m, C_m, \tilde{Q}) = \text{tr}\big[\tilde{Q}\tilde{R} + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V})\tilde{P}\big], \quad (4.49)$$

where the symmetric matrix $\tilde{P}$ is a Lagrange multiplier, $\tilde{Q}$ satisfies (4.20), $\tilde{A}$, $\tilde{R}$, and $\tilde{V}$ are defined in Section 4.2. Setting $\partial L/\partial\tilde{Q} = 0$ gives (4.23); $\tilde{Q}$ and $\tilde{P}$ are symmetric positive definite and can be partitioned as in (4.22) and (4.24). A straightforward calculation shows

$$\begin{aligned} \frac{\partial L}{\partial(A_m)_{ij}} &= 2\big(P_{12}^TQ_{12} + P_2Q_2\big)_{ij}, \quad (i,j) \in \mathcal{S}, \\ \frac{\partial L}{\partial B_m} &= 2\big(P_{12}^TB + P_2B_m\big)V, \\ \frac{\partial L}{\partial(C_m)_{ij}} &= 2\frac{\partial}{\partial(C_m)_{ij}}\big[\ \text{tr}\ (-Q_{12}^TC^TRC_m) \\ &\qquad + \text{tr}\ (Q_2C_m^TRC_m)\big] \\ &= 2R\big(C_mQ_2 - CQ_{12}\big)_{ij}, \quad i > 1. \end{aligned} \quad (4.50)$$

Let

$$A(\lambda) = A, \quad B(\lambda) = \lambda B + (1-\lambda)B_i,$$
$$C(\lambda) = \lambda C + (1-\lambda)C_i,$$

where B_i and C_i play the same role as in Section 4.1. Let

$$H_{A_m}(\lambda, x) = \frac{1}{2}\frac{\partial L}{\partial A_m} = (P_{12}^T Q_{12} + P_2 Q_2),$$

$$H_{B_m}(\lambda, x, B_i) = \frac{1}{2}\frac{\partial L}{\partial B_m} = (P_{12}^T B(\lambda) + P_2 B_m)V,$$

$$H_{C_m}(\lambda, x, C_i) = \frac{1}{2}\frac{\partial L}{\partial C_m} = R(C_m Q_2 - C(\lambda)Q_{12}), \tag{4.51}$$

where in H_{A_m} only those elements corresponding to the independent variables of A_m are nonzero and

$$x \equiv \begin{pmatrix} (A_m)_{\mathcal{S}} \\ \text{Vec }(B_m) \\ \text{Vec }(C_m)_{\mathcal{T}\cdot} \end{pmatrix} \tag{4.52}$$

denotes the independent variables, $(A_m)_{\mathcal{S}}$ is a vector consisting of those elements in A_m with indices in the set $\mathcal{S}$, i.e.,

$$(A_m)_{\mathcal{S}} = \left((A_m)_{21}, (A_m)_{22}, \cdots, (A_m)_{n_m n_m}\right)^T,$$

$(C_m)_{\mathcal{T}}$ is the matrix obtained from rows $\mathcal{T}\cdot = \{2, \ldots, l\}$ of C_m.

The homotopy map $\rho(\lambda, x, a)$ for Ly's formulation is now defined as

$$F_1(\lambda, x, a) = \Big[H_{A_m}(\lambda, x)\Big]_{\mathcal{S}}, \tag{4.53}$$

$$F_2(\lambda, x, a) = \text{Vec}\,\Big[H_{B_m}(\lambda, x, B_i)\Big], \tag{4.54}$$

$$F_3(\lambda, x, a) = \text{Vec}\,\Big[H_{C_m}(\lambda, x, C_i)\Big]_{\mathcal{T}\cdot}, \tag{4.55}$$

where again the subscripts $\mathcal{S}$ and $\mathcal{T}\cdot$ select the appropriate matrix elements, and

$$a \equiv \begin{pmatrix} \text{Vec }(B_i) \\ \text{Vec }(C_i) \end{pmatrix} \tag{4.56}$$

denotes the parameter variables. As discussed in Section 4.2, without loss of generality set $V = I$ in (4.54) and $R = I$ in (4.55).

The Jacobian matrix $D\rho(\lambda, x, a)$ of $\rho(\lambda, x, a)$ is

$$\begin{pmatrix} D_\lambda F_1 & D_x F_1 & 0 & 0 \\ D_\lambda F_2 & D_x F_2 & D_{B_i} F_2 & 0 \\ D_\lambda F_3 & D_x F_3 & 0 & D_{C_i} F_3 \end{pmatrix}. \tag{4.57}$$

Lemma 4.6. Suppose $\text{rank}(D_x F_1) = n_m$. Then the Jacobian matrix (4.57) has full column rank for all $0 \le \lambda < 1$, i.e., the homotopy map (4.53)–(4.55) is transversal to zero for all $0 \le \lambda < 1$.

Proof. A similar proof to that in Section 4.2 yields

$$\text{rank}(D_{B_i} F_2) = m n_m \quad \text{for} \quad \lambda \ne 1. \tag{4.58}$$

Ordering C_i and F_3 by rows gives

$$\begin{aligned} D_{C_i} F_3(\lambda, \theta, a) &= D_{C_i}(-C(\lambda)Q_{12})_{\mathcal{T}\cdot} \\ &= (\lambda - 1) D_{C_i}(C_i Q_{12})_{\mathcal{T}\cdot} \\ &= (\lambda - 1) D_{C_i}[(C_i)_{\mathcal{T}\cdot} Q_{12}] \\ &= (1-\lambda)\overbrace{\begin{pmatrix} 0 & Q_{12}^T & 0 & \cdots & 0 \\ 0 & 0 & Q_{12}^T & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{12}^T \end{pmatrix}}^{l \text{ times}}, \end{aligned} \tag{4.59}$$

and then as before

$$\text{rank}(D_{C_i} F_3) = (l-1) n_m \quad \text{for} \quad \lambda \ne 1. \tag{4.60}$$

Note that

$$\text{rank}(D_x F_1) = n_m,$$

which completes the proof. Q. E. D.

Note that there are only n_m components in F_1 but $(l+m)n_m + 1$ independent variables in x and λ. As $l + m \gg 1$ usually in real problems which have been considered previously (Ge *et al.*, 1996), all Jacobian matrices of F_1 in those problems satisfied the full rank condition. Since each of Q_{12}, P_{12}, Q_2, and P_2 are implicit functions of x and $A(\lambda)$, and one can not give explicit expressions for $D_x F_1$ or $D_{A_i} F_1$ as in (4.59) for $D_{C_i} F_3$ (which show clearly the rank conditions), it was necessary to *assume* that $\text{rank}(D_x F_1) = n_m$ in Lemma 4.6. To guarantee the full rank of $D\rho$ without this assumption, instead of using (4.53), let $x = (\eta, \zeta)$, $\eta \in \mathbf{E}^{n_m}$,

$$F_1(\lambda, x, a) = \lambda\Big[H_{A_m}(\lambda, x)\Big]_{\mathcal{S}} + (1-\lambda)(\eta - \eta_0), \tag{4.61}$$

with n_m independent parameter variables in η_0, which gives

$$D_{\eta_0} F_1 = (1-\lambda) I_{n_m} \quad \text{for} \quad \lambda \ne 1. \tag{4.62}$$

Combining (4.58), (4.60), and (4.62) completes the proof that the map (4.61), (4.54), and (4.55) is transversal to zero. Note that the homotopy construction in (4.61) is a theoretical convenience, and in practice the choice (4.53) has been entirely satisfactory.

5. BOUNDEDNESS OF $\rho_a^{-1}(0)$ FOR H^2 OPTIMAL MODEL ORDER REDUCTION PROBLEM

5.1. *Counterexample for optimal projection homotopies*

The zero set $\rho_a^{-1}(0)$ of a given homotopy map based on the optimal projection equations (4.1)–(4.3) is not always bounded, as shown by the following 2-dimensional example.

The system (Kabamba, 1985b) is given by

$$A = \begin{pmatrix} -0.25 & -0.4 \\ -0.4 & -0.72 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix}, \quad C = (1 \quad 1.2). \tag{5.1}$$

For the system (2.1)–(2.4) defined by (5.1), the solution set of the optimal projection equations (4.1)–(4.3) contains an isolated solution and a one-dimensional manifold of solutions.

The isolated solution of this system is

$$A_m = (-0.838521), \quad B_m = (1.537575), \quad C_m = (1.537575),$$

which was obtained by both POLSYS from HOMPACK (Watson *et al.*, 1987) and by a homotopy approach (Žigić *et al.*, 1993b). The one-dimensional manifold of solutions can be derived directly from equations (4.1)–(4.3) as follows.

Let $W_1 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $U_1 = (u_1, u_2)$, $\Sigma = \sigma$, $V = I$, and $R = I$. The optimal projection equations (4.1)–(4.3) for this problem can be written as

$$\begin{aligned}
0 &= -0.25w_1^2u_1\sigma - 0.4w_1w_2u_1\sigma - 0.4w_1^2u_2\sigma \\
&\quad - 0.72w_1w_2u_2\sigma - 0.25w_1\sigma - 0.4w_2\sigma \\
&\quad + u_1 + 1.2u_2, \\
0 &= -0.25w_1w_2u_1\sigma - 0.4w_2^2u_1\sigma - 0.4w_1w_2u_2\sigma \\
&\quad - 0.72w_2^2u_2\sigma - 0.4w_1\sigma - 0.72w_2\sigma \\
&\quad + 1.2u_1 + 1.44u_2, \qquad (5.2) \\
0 &= -0.25w_1u_1^2\sigma - 0.4w_2u_1^2\sigma - 0.4w_1u_1u_2\sigma \\
&\quad - 0.72w_2u_1u_2\sigma - 0.25u_1\sigma - 0.4u_2\sigma \\
&\quad + w_1 + 1.2w_2, \\
0 &= -0.25w_1u_1u_2\sigma - 0.4w_2u_1u_2\sigma - 0.4w_1u_2^2\sigma \\
&\quad - 0.72w_2u_2^2\sigma - 0.4u_1\sigma - 0.72u_2\sigma \\
&\quad + 1.2w_1 + 1.44w_2, \\
0 &= w_1u_1 + w_2u_2 - 1.
\end{aligned}$$

The triple (A_m, B_m, C_m) is given by

$$\begin{aligned}
A_m &= \Gamma A G^T = (u_1 \; u_2) \begin{pmatrix} -0.25 & -0.4 \\ -0.4 & -0.72 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\
&= w_1(-0.25u_1 - 0.4u_2) \\
&\quad + w_2(-0.4u_1 - 0.72u_2), \\
B_m &= \Gamma B = (u_1 \; u_2) \begin{pmatrix} 1 \\ 1.2 \end{pmatrix} = u_1 + 1.2u_2, \qquad (5.3) \\
C_m &= CG^T = (1 \quad 1.2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_1 + 1.2w_2,
\end{aligned}$$

where $\Gamma = U_1$ and $G = W_1^T$.

The zero set of (5.2) contains

$$\{(W_1, U_1, \Sigma) : w_1 = -1.2w_2, \quad u_1 = -1.2u_2, \quad u_2 = \frac{1}{2.44w_2}, \quad \sigma = 0\}$$

which is unbounded. Every point in this set corresponds to the same triple (A_m, B_m, C_m):

$$A_m = -.0491803, \quad B_m = 0, \quad C_m = 0.$$

The homotopy map based on the optimal projection equations is

$$\begin{aligned}
&U_1\, A(\lambda)\, W_1\, \Sigma\, W_1^T + \Sigma\, W_1^T\, A^T(\lambda) + U_1\, BVB^T = 0, \\
&A^T(\lambda)\, U_1^T\, \Sigma + U_1^T\, \Sigma\, U_1\, A(\lambda)\, W_1 + C^TRC\, W_1 = 0, \\
&U_1\, W_1 - I = 0, \qquad (5.4)
\end{aligned}$$

where $A(\lambda) = \lambda A + (1-\lambda)D$, and D is part of the parameter vector a in Theorem 3.3. The zero set $\rho_a^{-1}(0)$ of this homotopy map for the system (5.1) includes the subset

$$\{(\lambda, W_1, U_1, \Sigma) : \quad 0 \le \lambda < 1, \quad w_1 = -1.2w_2, \quad u_1 = -1.2u_2, \quad u_2 = \frac{1}{2.44w_2}, \quad \sigma = 0\}, \tag{5.5}$$

which is unbounded. This example shows that the zero set $\rho_a^{-1}(0)$ of a homotopy map can be unbounded and yet some zero curves may still converge to isolated solutions.

Note that, in practice, the algorithm in (Žigić *et al.*, 1993b) always maintains $\operatorname{rank}(\Sigma) = n_m$, where $n_m = 1$ in the above example. Solutions with $\Sigma = 0$ in the above example never come into play. Boundedness of $\rho_a^{-1}(0)$ for the optimal projection equations (4.1)–(4.3) can indeed be guaranteed with more sophisticated mathematics, a slightly different homotopy map from the one used in practice, and complex arithmetic for the curve tracking. This is pursued in Section 5.3.

5.2. *Simplification and example for input normal form homotopy*

The following corollary is needed to simplify the homotopy map based on the input normal form formulation for the H^2 optimal model order reduction problem.

Corollary 5.1. Let $\tilde{A}_I$, $\tilde{R}_I$, $\tilde{V}_I$ be defined as in Section 4.2, partitioned as in (4.21), let A_m be stable, and $\tilde{Q}_I$ satisfy (4.20). To minimize (4.19) under the constraints (4.17) and (4.20), the following two Lagrangians are equivalent:

$$\begin{aligned}
&L_1(A_m, B_m, C_m, \Omega, \tilde{Q}_I, M_c, M_o, \tilde{P}_I) \\
&\quad = \operatorname{tr}\Big[\tilde{Q}_I\tilde{R}_I + \big(A_m + A_m^T + B_mVB_m^T\big)M_c \\
&\qquad + \big(A_m^T\Omega + \Omega A_m + C_m^TRC_m\big)M_o \\
&\qquad + \big(\tilde{A}_I\tilde{Q}_I + \tilde{Q}_I\tilde{A}_I^T + \tilde{V}_I\big)\tilde{P}_I\Big], \qquad (5.6)
\end{aligned}$$

where the symmetric matrices M_o, M_c, and $\tilde{P}_I$ are Lagrange multipliers introduced in Section 4.2, and

$$L_2(A_m, B_m, C_m, \tilde{Q}_I, \tilde{P}_I) = \mathrm{tr}\Big[\tilde{Q}_I\tilde{R}_I + (\tilde{A}_I\tilde{Q}_I + \tilde{Q}_I\tilde{A}_I^T + \tilde{V}_I)\tilde{P}_I\Big], \qquad (5.7)$$

where $\tilde{Q}_I$ is restricted to the form

$$\tilde{Q}_I = \begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^T & I_{n_m} \end{pmatrix},$$

the Lagrange multiplier $\tilde{P}_I$ is restricted to the form

$$\tilde{P}_I = \begin{pmatrix} \bar{P}_1 & \bar{P}_{12} \\ \bar{P}_{12}^T & \Omega \end{pmatrix},$$

and $\Omega = \mathrm{diag}(\omega_1, \cdots, \omega_{n_m})$ is a positive definite matrix.

Proof. The proof is straightforward. Setting $\partial L/\partial\tilde{Q}_I = 0$ gives the same equation

$$\tilde{A}_I^T\tilde{P}_I + \tilde{P}_I\tilde{A}_I + \tilde{R}_I = 0 \qquad (5.8)$$

in both cases. Expanding (4.20) and (5.8) yields the equations for $\bar{Q}_2$ and $\bar{P}_2$. In the first case

$$A_m\bar{Q}_2 + \bar{Q}_2A_m^T + B_mVB_m^T = 0,$$
$$A_m^T\bar{P}_2 + \bar{P}_2A_m + C_m^TRC_m = 0.$$

Since the constraints (4.17) and (4.20) should be satisfied and A_m is stable, it follows that at a constrained minimum

$$\bar{Q}_2 = I_{n_m}, \qquad \bar{P}_2 = \Omega.$$

Q. E. D.

The partial derivatives $\dfrac{\partial L_2}{\partial B_m}$ and $\dfrac{\partial L_2}{\partial C_m}$ of L_2 can be computed as

$$\frac{\partial L_2}{\partial B_m} = 2(\bar{P}_{12}^TB + \Omega B_m)V,$$
$$\frac{\partial L_2}{\partial C_m} = 2R(C_m - C\bar{Q}_{12}).$$

The corresponding homotopy map (4.30) and (4.31) is now simplified as

$$\rho(\lambda, x, a) = \begin{pmatrix} \mathrm{Vec}\,(H_{B_m}(\lambda, x, a)) \\ \mathrm{Vec}\,(H_{C_m}(\lambda, x, a)) \end{pmatrix},$$

where

$$H_{B_m}(\lambda, x, a) = (\bar{P}_{12}^TB(\lambda) + \Omega B_m)V,$$
$$H_{C_m}(\lambda, x, a) = R(C_m - C(\lambda)\bar{Q}_{12}).$$

The zero set $\rho_a^{-1}(0)$ of a homotopy map based on the input normal form formulation given by (Ge *et al.*, 1994) is not always bounded, as shown by the following 2-dimensional example.

The system is given by

$$A = \begin{pmatrix} -0.895116 & 0.612237 \\ 0.612237 & -0.447393 \end{pmatrix}, \; B = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$
$$C = (-2 \quad 1). \qquad (5.9)$$

According to (Ge *et al.*, 1994), the initial point and the triple $(A(\lambda), B(\lambda), C(\lambda))$ are chosen as follows:

1) Transform the given triple (A, B, C) to balanced form (A_b, B_b, C_b), such that $A_b = T^{-1}AT$, $B_b = T^{-1}B$, and $C_b = CT$ satisfy

$$0 = A_b\Lambda + \Lambda A_b^T + B_bVB_b^T,$$
$$0 = A_b^T\Lambda + \Lambda A_b + C_b^TRC_b,$$

with a positive definite matrix $\Lambda = \mathrm{diag}\,(d_1, d_2, \cdots, d_n)$, $d_i \geq d_{i+1}$.

The balanced form of (5.9) is

$$A_b = \begin{pmatrix} -0.25297 & -0.5 \\ -0.5 & -1.0896 \end{pmatrix}, \; B_b = \begin{pmatrix} -1.232 \\ -1.866 \end{pmatrix},$$
$$C_b = (-1.232 \quad -1.866),$$

with

$$T = \begin{pmatrix} 0.866 & 0.5 \\ 0.5 & -0.866 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1.5978 \end{pmatrix}.$$

2) For $n_m = 1$, the parametrization $(A(\lambda), B(\lambda), C(\lambda))$ is chosen as

$$A(\lambda) = \lambda A + (1-\lambda)A_i = \begin{pmatrix} a_1(\lambda) & a_2(\lambda) \\ a_2(\lambda) & a_3(\lambda) \end{pmatrix}$$
$$= \begin{pmatrix} -0.6422\lambda - 0.25297 & 0.612237\lambda \\ 0.612237\lambda & 0.6431\lambda - 1.0896 \end{pmatrix},$$
$$B(\lambda) = \lambda B + (1-\lambda)B_i = \begin{pmatrix} b_1(\lambda) \\ b_2(\lambda) \end{pmatrix}$$
$$= \begin{pmatrix} -1.232 - 0.768\lambda \\ \lambda \end{pmatrix},$$
$$C(\lambda) = \lambda C + (1-\lambda)C_i = (\,c_1(\lambda) \quad c_2(\lambda)\,)$$
$$= (\,-1.232 - 0.768\lambda \quad \lambda\,) = B^T(\lambda).$$

where

$$A_i = \begin{pmatrix} -0.25297 & 0 \\ 0 & -1.0896 \end{pmatrix},$$
$$B_i = \begin{pmatrix} -1.232 \\ 0 \end{pmatrix}, \quad C_i = (-1.232 \quad 0).$$

For brevity, $a_1(\lambda)$, $a_2(\lambda)$, $a_3(\lambda)$, $b_1(\lambda)$, $b_2(\lambda)$, $c_1(\lambda)$, and $c_2(\lambda)$ will be denoted by a_1, a_2, a_3, b_1, b_2, c_1, and c_2 respectively in the following. As discussed in Section 4.2, without loss of generality, set $V = I$ and $R = I$.

For any $0 < \lambda < 1$, $B_m \in \mathbf{R}, B_m \neq 0$, let

$$A_m = \frac{-B_m^2}{2}, \quad C_m = -\sqrt{\Omega}B_m,$$
$$\bar{P}_2 \equiv \Omega = \Big[\frac{M(b_1 - b_2)b_1}{a_1 + A_m - Ma_2}\Big]^2,$$
$$M = \frac{a_2b_1 - b_2(b_1 - A_m)}{b_1(a_3 + A_m) - b_2a_2},$$
$$(\bar{P}_{12})_{12} = \frac{C_m(a_2b_1 - a_1b_2 - A_mb_2)}{a_2^2 - (a_1 + A_m)(a_3 + A_m)},$$
$$(\bar{P}_{12})_{11} = \frac{b_2C_m - (\bar{P}_{12})_{12}(a_3 + A_m)}{a_2},$$

$$(\bar{Q}_{12})_{11} = \frac{(\bar{P}_{12})_{11}}{\sqrt{\Omega}}, \quad (\bar{Q}_{12})_{12} = \frac{(\bar{P}_{12})_{12}}{\sqrt{\Omega}}.$$

Then

$$\rho(\lambda, x, a) = 0,$$
$$\tilde{A}_I(\lambda)\tilde{Q}_I + \tilde{Q}_I\tilde{A}_I^T(\lambda) + \tilde{V}_I(\lambda) = 0,$$
$$\tilde{A}_I^T(\lambda)\tilde{P}_I + \tilde{P}_I\tilde{A}_I(\lambda) + \tilde{R}_I(\lambda) = 0$$

are satisfied. The zero set $\rho_a^{-1}(0)$ of this homotopy map includes

$$\{(\lambda, B_m, C_m) : 0 < \lambda < 1, C_m = -\sqrt{\Omega}B_m\}. \tag{5.10}$$

Clearly, (5.10) is unbounded. If $B_m \neq 0$, then A_m is stable, (A_m, B_m) is controllable, and (A_m, C_m) is observable.

5.3. *Homogeneous transformation to avoid solutions at infinity*

As shown by the examples in Sections 5.1 and 5.2, the polynomial systems (4.1)–(4.3) or (4.30)–(4.31) may have solutions at infinity, and $\rho_a^{-1}(0)$ contains paths that diverge to infinity as λ approaches 1. Solutions at infinity can be avoided via the following transformation (Morgan and Sommese, 1989), (Morgan and Sommese, 1987a), which will be used in Section 5.4.

Let $f(z) = 0$ be a polynomial system of N equations in N unknowns, where $z \in \mathbf{C}^N$, and define $f'(z')$ as the homogenization of $f(z)$:

$$f'_j(z') = z_0^{d_j} f_j(z_1/z_0, \ldots, z_N/z_0), \quad j = 1, \ldots, N, \tag{5.11}$$

where $d_j = \deg(f_j)$. $f'(z') = 0$ is a system of N homogeneous equations in $N+1$ unknowns.

Note that, if $f'(z^0) = 0$, then $f'(cz^0) = 0$ for any complex scalar c. Therefore, we may take "solutions" of $f'(z') = 0$ to be (complex) lines through the origin in $\mathbf{C}^{N+1}$. The set of these lines is called complex projective space, denoted by $\mathbf{P}^N$, a smooth compact N-complex-dimensional manifold. It is natural to view $\mathbf{P}^N$ as a disjoint union of points $[(z_0, \ldots, z_N)]$ with $z_0 \neq 0$ and the "points at infinity", the points $[(z_0, \ldots, z_N)]$ with $z_0 = 0$. The solutions of $f'(z') = 0$ in $\mathbf{P}^N$ are identified with the solutions and solutions at infinity of $f(z) = 0$ as follows.

First, the solutions to $f(z) = 0$ can be identified with the solutions to $f'(z') = 0$ with $z_0 \neq 0$. Explicitly, if $L \in \mathbf{P}^N$ is a solution to $f'(z') = 0$, and $z' \in L$, with $z' = (z_0, \ldots, z_N)$ and $z_0 \neq 0$, then $z = (z_1/z_0, z_2/z_0, \ldots, z_N/z_0)$ is a solution to $f(z) = 0$. On the other hand, if $z \in \mathbf{C}^N$ is a solution to $f(z) = 0$, then the line through $z' = (1, z)$ is a solution to $f'(z') = 0$ with $z_0 = 1 \neq 0$. A "solution to $f(z) = 0$ at infinity" is simply a solution to $f'(z') = 0$ (in $\mathbf{P}^N$) generated by z' with $z_0 = 0$.

Define a homotopy map (in $\mathbf{P}^N$)

$$h(z', \lambda) = (1-\lambda)\gamma g'(z') + \lambda f'(z'), \tag{5.12}$$

where g' is a homogeneous system of N polynomials in $N+1$ variables, and γ is a randomly chosen complex number. Intuitively, let g' be chosen so that its homogeneous structure matches that of f'. Precisely, let $S \in \mathbf{P}^N$ be the set of common solutions of $f'(z') = 0$ and $g'(z') = 0$. Then for each $s \in S$ the following conditions must hold. For $s \in S$ let K denote the full connected component of solutions of $g'(z') = 0$ with $s \in K$.

If s is a geometrically isolated solution of $g'(z') = 0$, assume that: a) s is also a geometrically isolated solution of $f'(z') = 0$, and b) the multiplicity of s as a solution of $g'(z') = 0$ is less than or equal to the multiplicity of s as a solution of $f'(z') = 0$.

If s is not a geometrically isolated solution of $g'(z') = 0$, assume that: a) K is contained in S, b) K is the full solution component of $f'(z') = 0$ containing s, c) K is a smooth manifold, and d) at each point $z^0 \in K$ the rank of $\nabla g'(z^0)$ is the codimension of K.

Let S' denote the solution set of $g'(z') = 0$ in $\mathbf{P}^N - S$. Under these assumptions, the basic result is the following theorem.

Theorem 5.2 (Morgan and Sommese, 1987b). Assume the points in S' are nonsingular solutions of $g'(z') = 0$. For any positive r and for all but a finite number of angles θ, if $\gamma = re^{i\theta}$, then $h^{-1}(0) \cap ((\mathbf{P}^N - S) \times [0, 1))$ consists of smooth paths and every geometrically isolated solution of $f'(z') = 0$ not in S has a path in $(\mathbf{P}^N - S) \times [0, 1)$ converging to it.

Let

$$L(z') = \sum_{i=0}^{N} b_i z_i,$$

where $b_i \neq 0$ for some i.

$$U_L = \{[z'] \in \mathbf{P}^N \mid L(z') \neq 0\}$$

is the Euclidean coordinate patch on $\mathbf{P}^N$ defined by L. Note that U_L, which is an open dense submanifold of $\mathbf{P}^N$, can be identified with $\mathbf{C}^N$ via

$$[(z_0, \ldots, z_N)] \to \frac{1}{L(z')}(z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_N),$$

where $b_i \neq 0$.

The following theorem from (Morgan and Sommese, 1987b) shows how to keep the homotopy process in complex Euclidean space, even though the basic theorem is formulated in $\mathbf{P}^N$.

Theorem 5.3. Assume the points in S' are nonsingular solutions of $g'(z') = 0$. Then

$$\overline{h^{-1}(0) \cap ((\mathbf{P}^N - S) \times [0,1))} \subset U_L \times [0,1],$$

for almost all U_L and all but a finite number of angles θ.

For computations, the coordinate patch U_L is realized via a projective transformation as follows. With homogeneous h in the variables z_i for $i = 0$ to N, let

$$z_0 = \sum_{i=1}^{N} \beta_i z_i + \beta_0, \tag{5.13}$$

where the β_i are constants and $\beta_i \neq 0$ for all i. The projective transformation of h is the system H of N polynomials in the N variables z_i for $i = 1$ to N where $H_j = h_j$, with (5.13) defining z_0 in terms of the other variables. By Theorem 5.3, the homotopy paths, including end points, are completely represented in $\mathbf{C}^N$ via H. The finite solutions of $f(x) = 0$ are recovered via $z_i \leftarrow z_i/z_0$ for $i = 1$ to N. If $z_0 = 0$, then the solution is at infinity. This concludes the background discussion of polynomial system theory.

5.4. *Homogeneous transformation of optimal projection homotopies*

In this section the homogeneous transformation introduced in Section 5.3 is used to prevent unbounded zero sets for optimal projection homotopies. Consider the polynomial system given by (4.1)–(4.3) and the corresponding optimal projection homotopies defined in Section 4.1. The start system at $\lambda = 0$ is taken as

$$U_1 A(0) W_1 \Sigma W_1^T + \Sigma W_1^T A(0)^T + U_1 B(0) = 0,$$
$$A(0)^T U_1^T \Sigma + U_1^T \Sigma U_1 A(0) W_1 + C(0) W_1 = 0,$$
$$U_1 W_1 - I_{n_m} = 0, \tag{5.14}$$

where $A(0) = D = A - \epsilon I_n$, ϵ is a constant, $A(\lambda) = \lambda A + (1 - \lambda) D$. The target system (at $\lambda = 1$) is (4.1)-(4.3).

According to Section 4.3, the homogenization of the target system (4.1)–(4.3) can be taken as

$$U_1' A W_1' \Sigma' W_1'^T + z_0^2 \Sigma' W_1'^T A^T + z_0^3 U_1' B V B^T = 0,$$
$$z_0^2 A^T U_1'^T \Sigma' + U_1'^T \Sigma' U_1' A W_1' + z_0^3 C^T R C W_1' = 0,$$
$$U_1' W_1' - z_0^2 I_{n_m} = 0, \tag{5.15}$$

where $z = (\mathrm{vec}(U_1), \mathrm{vec}(W_1), \mathrm{vec}(\Sigma))$,

$$U_1'(z_0, \ldots, z_N) = z_0 U_1(z_1/z_0, \ldots, z_N/z_0),$$
$$W_1'(z_0, \ldots, z_N) = z_0 W_1(z_1/z_0, \ldots, z_N/z_0),$$
$$\Sigma'(z_0, \ldots, z_N) = z_0 \Sigma(z_1/z_0, \ldots, z_N/z_0).$$

The corresponding homogenization of the start system is

$$U_1' D W_1' \Sigma' W_1'^T + z_0^2 \Sigma' W_1'^T D^T + z_0^3 U_1' B_i = 0,$$
$$z_0^2 D^T U_1'^T \Sigma' + U_1'^T \Sigma' U_1' D W_1' + z_0^3 C_i W_1' = 0,$$
$$U_1' W_1' - z_0^2 I_{n_m} = 0, \tag{5.16}$$

where $B_i = B(0)$ and $C_i = C(0)$.

Theorem 5.4. If B_i, C_i, and ϵ can be chosen such that (5.16) and (5.15) have no common $z_0 \neq 0$, $\Sigma' \neq 0$ solutions, and all $z_0 \neq 0$, $\Sigma' \neq 0$ solutions of (5.16) are nonsingular, then every geometrically isolated solution of (5.15) has a path in $\mathbf{P}^N$ converging to it.

Proof. If $\epsilon = 0$, (5.15) and (5.16) have the same $z_0 = 0$ solution set (corresponding to solutions of (4.1)–(4.3) at infinity). Since B_i and C_i can be chosen such that (5.15) and (5.16) have no common $z_0 \neq 0$, $\Sigma' \neq 0$ solutions and all $z_0 \neq 0$, $\Sigma' \neq 0$ solutions of (5.16) are nonsingular, then all the conditions of Theorems 5.2 and 5.3 hold. For each point in S', the associated path in $H^{-1}(0)$ can be tracked from $\lambda = 0$ to $\lambda = 1$. This will yield the full list of geometrically isolated solutions to $H(z, 1) = 0$. No paths diverge to infinity.

If $\epsilon \neq 0$, $B(\lambda) = BVB^T$, and $C(\lambda) = C^T RC$ for $0 \leq \lambda \leq 1$ as in (Žigić *et al.*, 1993b), using the fact $U_1' W_1' = 0$ (when $z_0 = 0$), it is clear that the $z_0 = 0$ solution set of (5.16) is the same as that of (5.15). Similarly, (5.15) and (5.16) have the same $z_0 \neq 0$ solutions when $\Sigma' = 0$. Note that this case corresponds to the counterexample of Section 5.1. Take S be all the $z_0 = 0$ solutions and any solutions corresponding to $z_0 \neq 0$ and $\Sigma' = 0$. Now ϵ can be chosen such that (5.15) and (5.16) have no other common solutions and all other $z_0 \neq 0$ solutions of (5.16) are nonsingular. Then the technical assumptions of Theorem 5.2 can clearly be met for the common solution set S. Thus Theorem 5.2 and Theorem 5.3 hold for the start system (5.16) in this case ($\epsilon \neq 0$) also. Q. E. D.

The import of this result is that the real solutions of (4.1)–(4.3), which satisfy the rank condition

$$\mathrm{rank}(W_1) = \mathrm{rank}(U_1) = \mathrm{rank}(\Sigma) = n_m,$$

if they exist, must be connected to the solutions of (5.16) in $\mathbf{P}^N - S$. Technically, this is guaranteed only with a complex multiplier γ in (5.16), and only if complex arithmetic is used and the homotopy curve tracking is done in $\mathbf{P}^N$. However, all this has never been necessary in practice (Žigić *et al.*, 1993b). Furthermore, observe that the solution set (5.15) includes all solutions with rank $\Sigma' \leq n_m$, and thus one is guaranteed of finding a reduced order model of order *no greater than* n_m. Since (5.15) represents the optimal projection equations (4.1)–(4.3) for some stable $A(\lambda)$ for every λ, $0 \leq \lambda \leq 1$, it is clear why real arithmetic suffices generically. Generically, the real solutions are isolated, have

constant rank, and vary smoothly with respect to λ (Morgan and Sommese, 1989).

Finally, for the target system (5.15), it is always possible to take the starting homogeneous system as

$$p_j z_j^4 - q_j z_0^4 = 0, \quad j = 1, \ldots, N, \tag{5.17}$$

where p_j and q_j are positive constants such that (5.17) has no common solution with (5.15). Since all solutions to (5.17) are nonsingular, all conditions of Theorem 5.2 and Theorem 5.3 are satisfied. The drawback is that the starting system (5.17) is totally unrelated to (5.15), requires complex arithmetic, and may take more steps to converge.

6. CONCLUSIONS

Probability-one homotopy methods were considered for the problem of H^2 model reduction. The crucial requirement of transversality was verified for several homotopy maps including the pseudogramian formulation of the optimal projection equations as well as variations based upon canonical forms. These results guarantee good numerical properties in the computational implementation of probability-one homotopy algorithms. Counterexamples to the boundedness requirement of probability-one homotopy theory were provided for the pseudogramian formulation of the optimal projection equations and for some formulations based upon canonical forms. Since a solution may not exist in any particular canonical form, these results are sharp for canonical forms, where unboundedness corresponds to nonexistence of solutions. However, for a reformulation of the pseudogramian optimal projection equations in complex projective space using homogeneous transformations, the boundedness assumption holds and thus global convergence of the homotopy algorithm to a solution (in complex projective space) is guaranteed. Both the genericity of real solutions and considerable computational experience (Žigić *et al.*, 1993b) indicate that real-valued homotopies are effective in practice and thus it is not necessary to track the homotopy zero curves in complex projective space.

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The Robust Fixed-Structure Control Toolbox

A Collection of Matlab Functions for Synthesizing Robust Fixed-Structure Controllers

Version 1.0

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Contents

Chapter 1

Introduction

The *Robust Fixed-Structure Control Toolbox* is an integrated collection of MATLAB functions that can be used to synthesize fixed-structure controllers that are optimal with respect to a given performance measure and at the same time satisfy stability and robustness constraints. The *Robust Fixed-Structure Control Toolbox* is designed to solve a large class of problems, including decentralized compensation, reduced order compensation, controller design for multiple plant configurations, and real parameter model uncertainty. The flexibility of the toolbox routines is due to the use of a decentralized static output feedback framework (Chapter 2) in problem formulation, which encompasses all of the above problems, and allows them to be solved with a common solution algorithm.

Once a control synthesis problem has been transformed into the decentralized static output feedback framework, the next problem is to optimize the free parameters in the controller with respect to one of several *performance criterion* that are included in the *Robust Fixed-Structure Control Toolbox* (Chapter 3). A modified quasi-Newton unconstrained optimization algorithm [4] is used to accomplish this.

Following the discussion on performance criterion are demonstrations of the transformation of standard fixed-structure control synthesis problems into the decentralized static output feedback framework, along with MATLAB sessions showing how these problems are set up and solved using the *Robust Fixed-Structure Control Toolbox* (Chapter 4).

A discussion of the homotopy algorithms utilized in this toolbox can be found in Chapter 5, followed by demonstrations of formulations for the model order reduction and controller synthesis problems (Chapter 6).

Descriptions and syntax for all core commands in the *Robust Fixed-Structure Control Toolbox* are provided in this document (Chapter 7). Also included are several supplementary functions, some related to automating the process of transforming standard synthesis problems into the decentralized static output feedback framework, and others which can be used for generating initial controllers of a given structure to optimize. These supplementary routines, while

not exhaustive or complete, can be used to solve several commonly occurring synthesis problems.

Chapter 2

Decentralized Static Output Feedback

This section reviews the decentralized static output feedback problem formulation [8, 3] for fixed-structure controller synthesis. For both continuous and discrete-time problems, consider the $(m+p+1)$-vector-input, $(m+p+1)$-vector-output decentralized system shown in Figure 2.1, and define

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad d = \begin{bmatrix} d_1 \\ \vdots \\ d_p \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ \vdots \\ e_p \end{bmatrix}, \tag{2.1}$$

Let the system G have the realization

$$G(s) \sim \left[\begin{array}{c|c|c|c} \mathcal{A} & \mathcal{B}_u & \mathcal{B}_d & \mathcal{B}_w \\ \hline \mathcal{C}_y & \mathcal{D}_{yu} & \mathcal{D}_{yd} & \mathcal{D}_{yw} \\ \hline \mathcal{C}_e & \mathcal{D}_{eu} & \mathcal{D}_{ed} & \mathcal{D}_{ew} \\ \hline \mathcal{C}_z & \mathcal{D}_{zu} & \mathcal{D}_{zd} & \mathcal{D}_{zw} \end{array}\right]. \tag{2.2}$$

2.1 Continuous-Time Decentralized Static Output Feedback

In a continous-time framework, the realization of G (2.2) represents the dynamics

$$\dot{x}(t) \quad = \quad \mathcal{A}x(t) + \mathcal{B}_u u(t) + \mathcal{B}_d d(t) + \mathcal{B}_w w(t), \tag{2.3}$$

with measurements

$$y(t) \quad = \quad \mathcal{C}_y x(t) + \mathcal{D}_{yu} u(t) + \mathcal{D}_{yd} d(t) + \mathcal{D}_{yw} w(t), \tag{2.4}$$

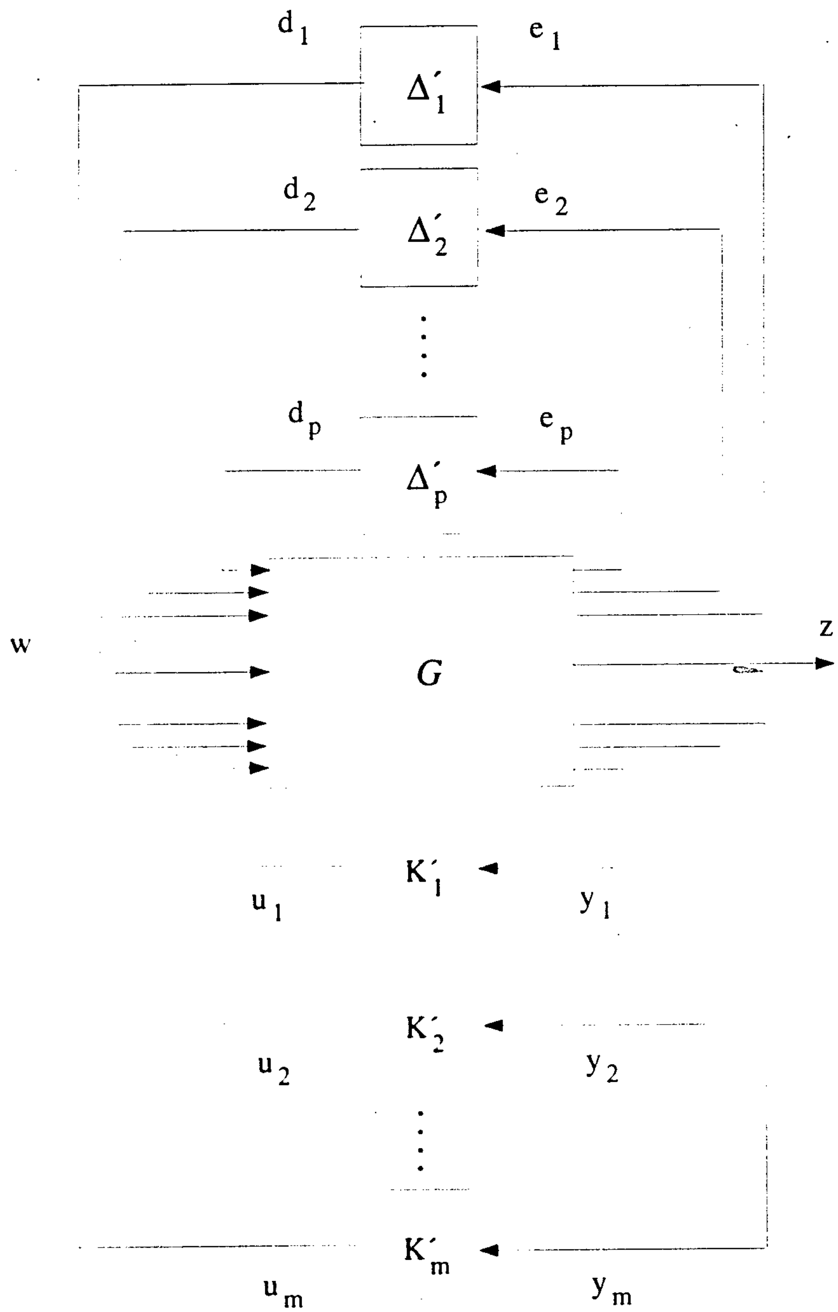

Figure 2.1: Decentralized Static Output Feedback Framework

model error outputs

$$e(t) \;=\; \mathcal{C}_e x(t) + \mathcal{D}_{eu} u(t) + \mathcal{D}_{ed} d(t) + \mathcal{D}_{ew} w(t), \tag{2.5}$$

and performance variables

$$z(t) \;=\; \mathcal{C}_z x(t) + \mathcal{D}_{zu} u(t) + \mathcal{D}_{zd} d(t) + \mathcal{D}_{zw} w(t). \tag{2.6}$$

Plant uncertainty is represented as decentralized static output feedback by the relations

$$d_i(t) = \Delta_i' e_i(t), \quad i = 1, \ldots, p, \tag{2.7}$$

where the uncertain matrices Δ_i' are not necessarily distinct. To represent decentralized static output feedback control with possibly repeated parameters, we consider

$$u_i(t) = \mathcal{K}_i' y_i(t), \quad i = 1, \ldots, m, \tag{2.8}$$

where the matrices $\mathcal{K}_i'$ are not necessarily distinct. Reordering the variables in (2.7) and (2.8) if necessary, we can rewrite (2.7), (2.8) as

$$d(t) \;=\; \Delta e(t), \tag{2.9}$$

$$u(t) \;=\; \mathcal{K} y(t), \tag{2.10}$$

where Δ and $\mathcal{K}$ have the form

$$\Delta \;\triangleq\; \text{block-diag}\left(I_{\psi_1} \odot \Delta_1, \ldots, I_{\psi_q} \odot \Delta_q\right), \tag{2.11}$$

$$\mathcal{K} \;\triangleq\; \text{block-diag}\left(I_{\phi_1} \odot \mathcal{K}_1, \ldots, I_{\phi_v} \odot \mathcal{K}_v\right), \tag{2.12}$$

where q is the number of *distinct* uncertainties $\Delta_i \in \mathcal{R}^{l_i \times g_i}$, ψ_i is the number of repetitions of uncertainty Δ_i, v is the number of *distinct* gains $\mathcal{K}_i \in \mathcal{R}^{r_i \times c_i}$ and ϕ_i is the number of repetitions of gain $\mathcal{K}_i$. Note that $\Delta_1, \ldots, \Delta_q$ and $\mathcal{K}_1, \ldots, \mathcal{K}_v$ are not necessarily square matrices, and that

$$\sum_{i=1}^{q} \psi_i = p, \qquad \sum_{i=1}^{v} \phi_i = m.$$

Alternatively, $\mathcal{K}$ can be written as

$$\mathcal{K} = \sum_{i=1}^{v} \sum_{j=1}^{\phi_i} Q_{\mathrm{L}ij} K_i Q_{\mathrm{R}ij}. \tag{2.13}$$

where $Q_{\mathrm{L}ij}$ and $Q_{\mathrm{R}ij}$ are defined as

$$Q_{\mathrm{L}ij} \triangleq \begin{bmatrix} 0_{r_1\phi_1\times r_i} \\ 0_{r_2\phi_2\times r_i} \\ \vdots \\ 0_{r_{i-1}\phi_{i-1}\times r_i} \\ 0_{r_i(j-1)\times r_i} \\ I_{r_i} \\ 0_{r_i(\phi_i-j)\times r_i} \\ 0_{r_{i+1}\phi_{i+1}\times r_i} \\ \vdots \\ 0_{r_v\phi_v\times r_i} \end{bmatrix} \qquad Q_{\mathrm{R}ij} \triangleq \begin{bmatrix} 0_{c_1\phi_1\times c_i} \\ 0_{c_2\phi_2\times c_i} \\ \vdots \\ 0_{c_{i-1}\phi_{i-1}\times c_i} \\ 0_{c_i(j-1)\times c_i} \\ I_{c_i} \\ 0_{c_i(\phi_i-j)\times c_i} \\ 0_{c_{i+1}\phi_{i+1}\times c_i} \\ \vdots \\ 0_{c_v\phi_v\times c_i} \end{bmatrix}^{\mathrm{T}} \tag{2.14}$$

For convenience, define

$$L_{\mathcal{K}} \triangleq I - \mathcal{D}_{yu}\mathcal{K}, \tag{2.15}$$

and assume that $L_{\mathcal{K}}$ is nonsingular. Furthermore, assume that

$$\mathcal{D}_{ed} = 0, \tag{2.16}$$

$$\mathcal{D}_{yd}\Delta\mathcal{D}_{eu} = 0 \text{ for all } \Delta \text{ of the form (2.11)}, \tag{2.17}$$

$$\mathcal{D}_{eu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yd} = 0 \text{ for all } \mathcal{K} \text{ of the form (2.12)}. \tag{2.18}$$

In this case, the closed-loop dynamics are

$$\dot{x}(t) = \left(\tilde{A} + \tilde{B}_d\Delta\tilde{C}_e\right)x(t) + \left(\tilde{B}_w + \tilde{B}_d\Delta\tilde{D}_{ew}\right)w(t), \tag{2.19}$$

$$z(t) = \left(\tilde{C}_z + \tilde{D}_{zd}\Delta\tilde{C}_e\right)x(t) + \left(\tilde{D}_{zw} + \tilde{D}_{zd}\Delta\tilde{D}_{ew}\right)w(t), \tag{2.20}$$

where

$$\begin{aligned} \tilde{A} &\triangleq \mathcal{A} + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y, & \tilde{B}_w &\triangleq \mathcal{B}_w + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw}, \\ \tilde{C}_z &\triangleq \mathcal{C}_z + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y, & \tilde{D}_{zw} &\triangleq \mathcal{D}_{zw} + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw}, \\ \tilde{B}_d &\triangleq \mathcal{B}_d + \mathcal{B}_u\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yd}, & \tilde{D}_{ew} &\triangleq \mathcal{D}_{ew} + \mathcal{D}_{eu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yw}, \\ \tilde{C}_e &\triangleq \mathcal{C}_e + \mathcal{D}_{eu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{C}_y, & \tilde{D}_{zd} &\triangleq \mathcal{D}_{zd} + \mathcal{D}_{zu}\mathcal{K}L_{\mathcal{K}}^{-1}\mathcal{D}_{yd}. \end{aligned} \tag{2.21}$$

The closed-loop transfer function $\tilde{G}_{zw,\Delta}(s)$ therefore has the realization

$$\tilde{G}_{zw,\Delta}(s) \sim \left[\begin{array}{c|c} \tilde{A} + \tilde{B}_d\Delta\tilde{C}_e & \tilde{B}_w + \tilde{B}_d\Delta\tilde{D}_{ew} \\ \hline \tilde{C}_z + \tilde{D}_{zd}\Delta\tilde{C}_e & \tilde{D}_{zw} + \tilde{D}_{zd}\Delta\tilde{D}_{ew} \end{array}\right]. \tag{2.22}$$

The nominal closed-loop transfer function $\tilde{G}_{zw}(s)$ is obtained by letting $\Delta = 0$, so that

$$\tilde{G}_{zw}(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_w \\ \hline \tilde{C}_z & \tilde{D}_{zw} \end{array}\right]. \tag{2.23}$$

2.2 Discrete-Time Decentralized Static Output Feedback

In a discrete-time framework, the realization of G (2.2) represents the dynamics

$$x(k+1) = \mathcal{A}x(k) + \mathcal{B}_u u(k) + \mathcal{B}_d d(k) + \mathcal{B}_w w(k), \tag{2.24}$$

with measurements

$$y(k) = \mathcal{C}_y x(k) + \mathcal{D}_{yu} u(k) + \mathcal{D}_{yd} d(k) + \mathcal{D}_{yw} w(k), \tag{2.25}$$

model error outputs

$$e(k) = \mathcal{C}_e x(k) + \mathcal{D}_{eu} u(k) + \mathcal{D}_{ed} d(k) + \mathcal{D}_{ew} w(k), \tag{2.26}$$

and performance variables

$$z(k) = \mathcal{C}_z x(k) + \mathcal{D}_{zu} u(k) + \mathcal{D}_{zd} d(k) + \mathcal{D}_{zw} w(k). \tag{2.27}$$

Plant uncertainty is represented as decentralized static output feedback by the relations

$$d_i(k) = \Delta_i' e_i(k), \quad i = 1, \ldots, p, \tag{2.28}$$

where again the uncertain matrices Δ_i' are not necessarily distinct. To represent decentralized static output feedback control with possibly repeated parameters, we consider

$$u_i(k) = \mathcal{K}_i' y_i(k), \quad i = 1, \ldots, m, \tag{2.29}$$

where the matrices $\mathcal{K}_i'$ are not necessarily distinct. Reordering the variables in (2.28) and (2.29) if necessary, we can rewrite (2.28), (2.29) as

$$d(k) = \Delta e(k), \tag{2.30}$$
$$u(k) = \mathcal{K} y(k), \tag{2.31}$$

where Δ and $\mathcal{K}$ have the forms (2.11) and (2.12), respectively. Thus, the alternative representation of $\mathcal{K}$ (2.13) still holds. Using the same definition of $L_{\mathcal{K}}$ (2.15), and making the same assumptions as in the continuous-time framework (2.16)–(2.18), the closed-loop dynamics are

$$x(k+1) = \left(\tilde{A} + \tilde{B}_d \Delta \tilde{C}_e\right) x(k) + \left(\tilde{B}_w + \tilde{B}_d \Delta \tilde{D}_{ew}\right) w(k), \tag{2.32}$$

$$z(k) = \left(\tilde{C}_z + \tilde{D}_{zd} \Delta \tilde{C}_e\right) x(k) + \left(\tilde{D}_{zw} + \tilde{D}_{zd} \Delta \tilde{D}_{ew}\right) w(k), \tag{2.33}$$

where we make use of the definitions (2.21). The discrete-time closed-loop transfer function $\tilde{G}_{zw,\Delta}(z)$ therefore has the realization

$$\tilde{G}_{zw,\Delta}(z) \sim \left[\begin{array}{c|c} \tilde{A} + \tilde{B}_d \Delta \tilde{C}_e & \tilde{B}_w + \tilde{B}_d \Delta \tilde{D}_{ew} \\ \hline \tilde{C}_z + \tilde{D}_{zd} \Delta \tilde{C}_e & \tilde{D}_{zw} + \tilde{D}_{zd} \Delta \tilde{D}_{ew} \end{array}\right], \tag{2.34}$$

and the nominal closed-loop transfer function $\tilde{G}_{zw}(z)$ has the realization

$$\tilde{G}_{zw}(z) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_w \\ \hline \tilde{C}_z & \tilde{D}_{zw} \end{array}\right]. \tag{2.35}$$

Chapter 3

Performance Criterion

The *Robust Fixed-Structure Control Toolbox* can be used to synthesize controllers that are optimal with respect to a user-chosen performance criterion. A performance criterion consists of a cost function and possibly one or more constraints. The cost function represents some characteristic of the controlled system, while the constraints represent properties that any feasible solution to the optimization problem must have. An example of a cost function is a norm on a closed-loop transfer function, while examples of constraints include asymptotic stability of the nominal closed-loop system or robust stability with respect to uncertainties of a certain size and structure.

The performance criterion options available in the *Robust Fixed-Structure Control Toolbox* are described in the following sections.

3.1 Continuous-Time $\mathcal{H}_2$-optimal Performance

Continuous-time $\mathcal{H}_2$-optimal performance is obtained by optimizing the $\mathcal{H}_2$ norm of the nominal closed-loop system(w to z) with respect to the free parameters of the controller. Since $\mathcal{H}_2$-optimal performance does not account for uncertainty in the model, poor or even destabilizing designs can result if the true plant differs from the nominal plant.

If $\tilde{G}_{zw}(s)$ is an asymptotically stable, strictly proper continuous-time transfer function with the realization

$$\tilde{G}_{zw}(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_w \\ \hline \tilde{C}_z & 0 \end{array}\right], \tag{3.1}$$

then it can be shown that $\| \tilde{G}_{zw}(s) \|_2$ can be expressed as

$$\| \tilde{G}_{zw}(s) \|_2^2 = \operatorname{tr} \tilde{P}\tilde{V}, \tag{3.2}$$

where $\tilde{P}$ is the solution of the continuous-time matrix Lyapunov equation

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \tilde{R}, \tag{3.3}$$

with $\tilde{R} = \tilde{C}_z^{\mathrm{T}}\tilde{C}_z$ and $\tilde{V} = \tilde{B}_w\tilde{B}_w^{\mathrm{T}}$.

The $\mathcal{H}_2$ norm is only defined for stable, strictly proper transfer functions. Thus, when using the $\mathcal{H}_2$-optimal performance criterion, the closed-loop system will be constrained to be asymptotically stable, and the closed-loop feed-through matrix $\tilde{D}_{zw}$ must be identically zero.

3.2 Discrete-Time $\mathcal{H}_2$-optimal Performance

Discrete-Time $\mathcal{H}_2$-optimal performance is obtained by optimizing the discrete-time $\mathcal{H}_2$ norm of the nominal closed-loop system (w to z) with respect to the free parameters of the controller gain matrices. As in the continuous-time case, discrete-time $\mathcal{H}_2$-optimal performance does not account for uncertainty in the model.

If $\tilde{G}_{zw}(z)$ is a proper, asymptotically stable discrete-time transfe r function with the realization

$$\tilde{G}_{zw}(z) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_w \\ \hline \tilde{C}_z & 0 \end{array}\right], \tag{3.4}$$

then it can be shown that $\| G(z) \|_2$ can be expressed as

$$\| \tilde{G}_{zw}(z) \|_2^2 = \operatorname{tr} \tilde{P}\tilde{V} + \tilde{D}_{zw}^{\mathrm{T}}\tilde{D}_{zw}, \tag{3.5}$$

s the solution of the discrete-time matrix Lyapunov equ ation

$$\tilde{P} = \tilde{A}^{\mathrm{T}}\tilde{P}\tilde{A} + \tilde{R}, \tag{3.6}$$

th $\tilde{R} = \tilde{C}_z^{\mathrm{T}}\tilde{C}_z$ and $\tilde{V} = \tilde{B}_w\tilde{B}_u^{\mathrm{T}}$.

The discrete-time $\mathcal{H}_2$ norm is only defined for asymptotically stable, proper transfer functions. Thus, the closed-loop system is constrained to be asymptotically stable when using the discrete-time $\mathcal{H}_2$-optimal performance criterion.

3.3 Scaled Popov Performance

Scaled Popov performance is characterized by robustness to uncertainty in the plant dynamics, input behavior, output behavior, or any combination thereof, which can be modeled by real but uncertain matrices inserted into the open-loop plant

$$\begin{aligned} \dot{x}(t) &= (A + M_A\Delta_A N_A)x(t) + (B_u + M_B\Delta_B N_B)u(t) + B_w w(t), \\ y(t) &= (C_y + M_C\Delta_C N_C)x(t) + (D_{yu} + M_D\Delta_D N_D)u(t) + D_{yw}w(t), \\ z(t) &= C_z x(t) + D_{zu}u(t) + D_{zw}w(t). \end{aligned} \tag{3.7}$$

To synthesize controllers which will exhibit robustness to these perturbations of the nominal plant, define Δ as

$$\Delta \triangleq \text{block-diag}(\Delta_A, \Delta_B, \Delta_C, \Delta D). \tag{3.8}$$

It is assumed that Δ is an element of the set of norm-bounded matrices $\mathbf{\Delta}_\gamma$, defined by

$$\mathbf{\Delta}_\gamma \triangleq \{\Delta \in \mathbf{\Delta} : \sigma_{\max}(\Delta) \leq \gamma^{-1}\} \tag{3.9}$$

where $\mathbf{\Delta}$ is the set of matrices with the specified internal structure of Δ. Chapter 4 gives examples of how to transform (3.7) into the equivalent decentralized static output feedback problem given by (2.2)

Scaled Popov performance produces controllers which guarantee closed-loop asymptotic stability for all Δ in the set $\mathbf{\Delta}_\gamma$, while at the same time minimizing a bound on the worst-case $\mathcal{H}_2$ norm of the closed-loop transfer function $\tilde{G}_{zw}(\jmath\omega)$ subject to the uncertainty Δ. Thus, the controllers not only have a priori regions of guaranteed stability, but also have a priori bounds on the $\mathcal{H}_2$ norm of the closed-loop system for all perturbations within the given class of perturbations $\mathbf{\Delta}_\gamma$.

The worst-case $\mathcal{H}_2$ norm for the closed-loop system from w to z subject to the perturbation Δ is given by

$$\sup_{\Delta \in \mathbf{\Delta}_\gamma} \| \tilde{G}_{zw,\Delta}(s) \|_2^2 = \sup_{\Delta \in \mathbf{\Delta}_\gamma} \operatorname{tr} \tilde{P}_\Delta \tilde{B}_w \tilde{B}_w^{\mathrm{T}}, \tag{3.10}$$

where $\tilde{P}_\Delta$ is the unique, nonnegative definite solution to the matrix Lyapunov equation

$$0 = (\tilde{A} + \tilde{B}_d \Delta \tilde{C}_e)^{\mathrm{T}} \tilde{P}_\Delta + \tilde{P}_\Delta (\tilde{A} + \tilde{B}_d \Delta \tilde{C}_e) + \tilde{C}_z^{\mathrm{T}} \tilde{C}_z. \tag{3.11}$$

Sparks and Bernstein [10] have shown that if

$$\tilde{A}_0 \triangleq \tilde{A} - \gamma^{-1} \tilde{B}_d \tilde{C}_e$$

is asymptotically stable, and there exists a positive-definite matrix $\tilde{P}$ and two matrices $Z > 0$, the *stability multiplier* matrix, and W, the *scaling* matrix which commute with the uncertainty Δ, such that

$$\Gamma \triangleq \gamma Z - W \tilde{C}_e \tilde{B}_d - \tilde{B}_d^{\mathrm{T}} \tilde{C}_e^{\mathrm{T}} W > 0, \tag{3.12}$$

and

$$0 = \tilde{A}_0^{\mathrm{T}} \tilde{P} + \tilde{P} \tilde{A}_0 + \mathcal{X}^{\mathrm{T}} \Gamma^{-1} \mathcal{X} + \tilde{R}, \tag{3.13}$$

where

$$\mathcal{X} \triangleq \tilde{B}_d^{\mathrm{T}} \tilde{P} + Z \tilde{C}_e + W \tilde{C}_e \tilde{A}_0, \tag{3.14}$$

then the closed-loop system is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\gamma$, and the worst-case $\mathcal{H}_2$ norm obeys the inequality

$$\sup_{\Delta \in \mathbf{\Delta}_\gamma} \| \tilde{G}_{zw,\Delta}(s) \|_2^2 \leq \operatorname{tr}(\tilde{P} + 2\gamma^{-1} \tilde{C}_e^{\mathrm{T}} W \tilde{C}_e) \tilde{B} \tilde{B}^{\mathrm{T}}. \tag{3.15}$$

Chapter 4

Formulation (Unconstrained Optimization)

In this section we introduce several controller synthesis problems, and provide the equivalent decentralized static output feedback problem. Unless specifically stated otherwise within a section, a plant of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + D_1 w, & (4.1)\\ y &= Cx + Fu + D_2 w, & (4.2)\\ z &= E_1 x + E_2 u + E_0 w, & (4.3) \end{aligned}$$

for continous-time problems or

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + D_1 w(k), & (4.4)\\ y(k) &= Cx(k) + Fu(k) + D_2 w(k), & (4.5)\\ z(k) &= E_1 x(k) + E_2 u(k) + E_0 w(k), & (4.6) \end{aligned}$$

for discrete-time problems is considered.

4.1 Centralized Proper Dynamic Compensation

First consider a full- or reduced-order proper, centralized dynamic compensator

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y, & (4.7)\\ u &= C_c x_c + D_c y, & (4.8) \end{aligned}$$

or, in discrete-time

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c y(k), & (4.9)\\ u(k) &= C_c x_c(k) + D_c y(k). & (4.10) \end{aligned}$$

Letting $\mathcal{K}$ denote the partitioned matrix

$$\mathcal{K} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}, \tag{4.11}$$

$L_{\mathcal{K}}$ is given by

$$L_{\mathcal{K}} = \begin{bmatrix} I & 0 \\ -FC_c & I - FD_c \end{bmatrix}.$$

Assuming the matrix $I - FD_c$ is nonsingular, it follows that $L_{\mathcal{K}}$ is nonsingular, and thus the closed-loop system consisting of (4.1)–(4.3), (4.7), and (4.8) (or (4.4)–(4.6), (4.9), and (4.10) in discrete-time) can be written as decentralized static output feedback with $m = r = \phi_1 = 1$ and G given by

$$G(s) \sim \left[\begin{array}{cc|cc} A & 0 & 0 & B|0|D_1 \\ 0 & 0 & I & 0|0|0 \\ \hline 0 & I & 0 & 0|0|0 \\ C & 0 & 0 & F|0|D_2 \\ \hline 0 & 0 & 0 & 0|0|0 \\ \hline E_1 & 0 & 0 & E_2|0|E_0 \end{array}\right]. \tag{4.12}$$

As an example, let us use the *Robust Fixed-Structure Control Toolbox* to design an $\mathcal{H}_2$-optimal compensator with this structure. First, we enter the standard control problem plant.

```
>> A = [zeros(5,1),eye(5);-1    -2   -24   -12   -24    -4];
>> B = [zeros(5,1);1];
>> C = [1 0 0 0 0 0];
>> D = 0;
>> D1 = [B,zeros(6,1)];
>> D2 = [0,1];
>> E1 = [C;zeros(1,6)];
>> E2 = [0;1];
>> E0 = zeros(2,2);
```

Next, we generate an initial guess for the optimal compensator by using a balanced truncation of the full-order $\mathcal{H}_2$-optimal compensator, via **rolqg**. We also create the matrices **QL11** and **QR11**, according to (2.14), which define the structure of $\mathcal{K}$.

```
>> nc = 4;
>> np = size(A,1);
>> [Ac,Bc,Cc,Dc,test] = rolqg(A,B,D1,C,E1,D,D2,E2,E0,nc);
>> k1 = [Ac,Bc;Cc,Dc]
k1 =
```

```
   -0.0157   -0.3137    0.0046   -0.0016   -0.0763
    0.3137   -0.3487    0.0275   -0.0093    0.2845
   -0.0046    0.0275   -0.2634    1.0135   -0.0112
   -0.0016    0.0093   -1.0135   -0.0360   -0.0038
   -0.0763   -0.2845    0.0112   -0.0038         0
>> QL11 = eye(5);
>> QR11 = eye(5);
>> save init k1 QL11 QR11
```

We now transform the standard control problem into the equivalent decentralized static output feedback framework problem via the realization (4.12).

```
>> A = [A,zeros(np,nc);zeros(nc,np),zeros(nc,nc)];
>> Bu = [zeros(np,nc),B;eye(nc),zeros(nc,1)];
>> Bw = [D1;zeros(nc,2)];
>> Cy = [zeros(nc,np),eye(nc);C,zeros(1,nc)];
>> Cz = [E1,zeros(2,nc)];
>> Dyu = [zeros(nc,nc),zeros(nc,1);zeros(1,nc),D];
>> Dyw = [zeros(nc,2);D2];
>> Dzu = [zeros(2,nc),E2];
>> Dzw = [E0];
>> clear B C D D1 D2 E1 E2 E0 nc np
>> who
Your variables are:
A          Bw         Cz         Dyw        Dzw
Bu         Cy         Dyu        Dzu
```

We now enter the remaining variables that are needed by optgain (see Chapter 7), and save them to a file

```
>> kindex = 1;
>> kindexkc = 5;
>> clear kindexkc
>> indexkc = 5;
>> indexkr = 5;
>> ctype = 1;
>> N = 1;
>> who
Your variables are:
A          Cy         Dyw        N          indexkr
```

```
Bu          Cz          Dzu         ctype       kindex
Bw          Dyu         Dzw         indexkc
>> save datafile
```

We load the file `init.mat` containing our initial value for the controller and `datafile.mat` containing the decentralized static output feedback problem data, and save the combined data to a single file, `runfile.mat`.

```
>> clear
>> load datafile
>> load init
>> who
Your variables are:
A           Cy          Dyw         N           ctype       k1
Bu          Cz          Dzu         QL11        indexkc     kindex
Bw          Dyu         Dzw         QR11        indexkr
>> save runfile
```

We now clear the workspace and execute `optgain`.

```
>> clear
>> [fmin,info,noits] = optgain('runfile','outfile')
 USING LINE SEARCH ALGORITHM
     ITERATE           FUNCTION VALUE
     -------           --------------
        0           0.2821656152736E+00
        1           0.2815935488427E+00
        2           0.2685207335632E+00
        3           0.2660287632697E+00
        4           0.2646826543299E+00
        5           0.2635998976307E+00
        6           0.2633244715048E+00
        7           0.2632156677085E+00
        8           0.2632052791173E+00
        9           0.2631913346650E+00
       10           0.2631636997435E+00
       11           0.2631177717331E+00
       12           0.2630841561395E+00
       13           0.2630724598839E+00
       14           0.2630685095928E+00
```

```
 TOTAL ITERATIONS =     14
 UNCMND WARNING -- INFO = 1: PROBABLY CONVERGED, GRADIENT SMALL
fmin =
    0.2631
info =
     1
noits =
    14
```

We load the file `outfile.mat` and examine the optimized controller parameters.

```
>> clear
>> load outfile
>> who
Your variables are:
fmin      info      kopt1
>> kopt1
kopt1 =
    0.0032   -0.2990    0.0067   -0.0019   -0.1655
    0.2990   -0.3610    0.0282   -0.0109    0.1538
   -0.0067    0.0282   -0.2634    1.0135   -0.0099
   -0.0019    0.0109   -1.0135   -0.0360    0.0028
   -0.1655   -0.1538    0.0099    0.0028   -0.2535
>>
```

4.2 Centralized Strictly Proper Dynamic Compensation

Consider a full- or reduced-order strictly proper, centralized dynamic compensator having the realization

$$\dot{x}_c = A_c x_c + B_c y, \quad (4.13)$$
$$u = C_c x_c. \quad (4.14)$$

Letting K denote the block-diagonal matrix

$$K = \text{block-diag}(A_c, B_c, C_c), \quad (4.15)$$

it can be verified that L_K is nonsingular. Hence, the closed-loop system consisting of (4.1)–(4.3) and (4.13)–(4.14) can be written as decentralized static

output feedback with $m = v = 3$, $\phi_1 = \phi_2 = \phi_3 = 1$, and $G(s)$ given by

$$G(s) \sim \left[\begin{array}{cc|cc|c|c|c} A & 0 & 0 & 0 & B & 0 & D_1 \\ 0 & 0 & I & I & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & F & 0 & D_2 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline E_1 & 0 & 0 & 0 & E_2 & 0 & E_0 \end{array}\right]. \tag{4.16}$$

We now demonstrate how to use the *Robust Fixed-Structure Control Toolbox* to design a continuous-time $\mathcal{H}_2$-optimal compensator with this structure. First, we enter the standard control problem plant,

```
>> A = [zeros(5,1),eye(5);-1    -2    -24    -12    -24     -4];
>> B = [zeros(5,1);1];
>> C = [1 0 0 0 0 0];
>> D = 0;
>> D1 = [B,zeros(6,1)];
>> D2 = [0,1];
>> E1 = [C;zeros(1,6)];
>> E2 = [0;1];
>> E0 = zeros(2,2);
```

Next, we generate an initial guess for the optimal compensator by using a balanced truncation of the full-order $\mathcal{H}_2$-optimal compensator, via `rolqg`. We also create the matrices QLij and QRij, according to (2.14), which define the structure of K.

```
>> nc = 1;
>> np = size(A,1);
>> [Ac,Bc,Cc,Dc,test] = rolqg(A,B,D1,C,E1,D,D2,E2,E0,nc);
>> k1 = Ac;
>> k2 = Bc;
>> k3 = Cc;
>> Dc
Dc =
     0
>> QL11 = [eye(nc);zeros(nc,nc);zeros(1,nc)];
>> QR11 = [eye(nc),zeros(nc,1),zeros(nc,nc)];
>> QL21 = [zeros(nc,nc);eye(nc);zeros(1,nc)];
>> QR21 = [zeros(1,nc),1,zeros(1,nc)];
```

```
>> QL31 = [zeros(nc,1);zeros(nc,1);1];
>> QR31 = [zeros(nc,nc),zeros(nc,1),eye(nc)];
>> save init k1 k2 k3 QL11 QR11 QL21 QR21 QL31 QR31
>> clear k1 k2 k3 QL11 QR11 QL21 QR21 QL31 QR31 Ac Bc Cc Dc test
```

We now transform the standard control problem into the equivalent decentralized static output feedback framework problem via the realization (4.16).

```
>> A = [A,zeros(np,nc);zeros(nc,np),zeros(nc,nc)];
>> Bu = [zeros(np,nc),zeros(np,nc),B;eye(nc),eye(nc),zeros(nc,1)];
>> Bw = [D1;zeros(nc,2)];
>> Cy = [zeros(nc,np),eye(nc);C,zeros(1,nc);zeros(nc,np),eye(nc)];
>> Cz = [E1,zeros(2,nc)];
>> Dyu = [zeros(nc,nc),zeros(nc,nc),zeros(nc,1);zeros(1,nc),zeros(1,nc),D;
   zeros(nc,nc),zeros(nc,nc),zeros(nc,1)];
>> Dyw = [zeros(nc,2);D2;zeros(nc,2)];
>> Dzu = [zeros(2,nc),zeros(2,nc),E2];
>> Dzw = E0;
>> clear B C D D1 D2 E1 E2 E0 nc np
>> who
Your variables are:
A          Bw         Cz         Dyw        Dzw
Bu         Cy         Dyu        Dzu
```

We now enter the remaining variables that are needed by **optgain** (see Chapter 7), and save them to a file

```
>> kindex = [1,1,1];
>> indexkc = [4,1,4];
>> indexkr = [4,4,1];
>> N = 3;
>> ctype = 1;
>> who
Your variables are:
A          Cy         Dyw        N          indexkr
Bu         Cz         Dzu        ctype      kindex
Bw         Dyu        Dzw        indexkc
>> save datafile
```

We load the file **init.mat** containing our initial value for the controller and **datafile.mat** containing the decentralized static output feedback problem data, and save the combined data to a single file, **runfile.mat**.

```
>> clear
>> load datafile
>> load init
>> who
Your variables are:
A        Cz       Dzw      QL31     ctype    k2
Bu       Dyu      N        QR11     indexkc  k3
Bw       Dyw      QL11     QR21     indexkr  kindex
Cy       Dzu      QL21     QR31     k1
>> save runfile
```

We now clear the workspace and execute `optgain`.

```
>> clear
>> [fmin,info,noits] = optgain('runfile','outfile')
 USING LINE SEARCH ALGORITHM
     ITERATE            FUNCTION VALUE
     -------            --------------
        0            0.4106206586956E+00
        1            0.4055239785049E+00
        2            0.3553510422910E+00
        3            0.3550592319726E+00
        4            0.3538665020519E+00
        5            0.3532465844085E+00
ATTEMPTED STABILITY CONSTRAINT VIOLATION
        6            0.3523475621202E+00
ATTEMPTED STABILITY CONSTRAINT VIOLATION
        7            0.3513753796696E+00
        :          :
       19            0.3385243413508E+00
       20            0.3331794494535E+00
       21            0.3246091693388E+00
       22            0.3194742011990E+00
       23            0.3168917100325E+00
       24            0.3157312171398E+00
       25            0.3153669369866E+00
       26            0.3153121011808E+00
       27            0.3153090709009E+00
```

```
        28              0.3153090381912E+00
 TOTAL ITERATIONS =     28
 UNCMND WARNING -- INFO = 1: PROBABLY CONVERGED, GRADIENT SMALL
fmin =
    0.3153
info =
     1
noits =
    28
```

Note the warning **ATTEMPTED STABILITY CONSTRAINT VIOLATION** appears whenever the optimization algorithm was required to reduce the step length in order to satisfy a constraint. This warning will appear occaisonally during normal optimization. Similar warnings appear during scaled Popov synthesis.

We now load the file **outfile.mat** and examine the optimized controller parameters.

```
>> clear
>> load outfile
>> who
Your variables are:
fmin      info      kopt1      kopt2      kopt3
>> kopt1
kopt1 =
   -0.0920
>> kopt2
kopt2 =
    0.1566
>> kopt3
kopt3 =
    0.1566
```

As a final note, the function **dsofformat** could have been used to transform the standard problem into the equivalent decentralized static output feedback framework problem; after entering the standard control problem plant and generating and saving an initial set of parameters (**k1**, **k2**, and **k3**), we execute **dsofformat** as follows

```
>> dsofformat('datafile',1,[1,1],A,B,D1,C,E1,D,D2,E2,E0);
>> clear
>> load datafile
>> who
```

```
Your variables are:

A         Cz        Dzw       QL31      indexkc
Bu        Dyu       N         QR11      indexkr
Bw        Dyw       QL11      QR21      kindex
Cy        Dzu       QL21      QR31
```

Note that **dsofformat** not only generates the decentralized static output feedback realization, but also the matrices **QL***ij* and **QR***ij*, and the variables **N**, **kindex**, **indexkc**, and **indexkr**.

4.3 Decentralized Strictly Proper Dynamic Compensation

Let u and y each be partitioned into two vector channels by $u = \begin{bmatrix} u_1^{\mathrm{T}} & u_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $y = \begin{bmatrix} y_1^{\mathrm{T}} & y_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, and rewrite (4.1)-(4.3) as

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 + D_1 w, \tag{4.17}$$
$$y_1 = C_1 x + F_{11} u_1 + F_{12} u_2 + D_{21} w, \tag{4.18}$$
$$y_2 = C_2 x + F_{21} u_1 + F_{22} u_2 + D_{22} w, \tag{4.19}$$
$$z = E_1 x + E_{21} u_1 + E_{22} u_2 + E_0 w. \tag{4.20}$$

We consider a two-channel, decentralized, strictly proper dynamic compensator

$$\dot{x}_{c1} = A_{c1} x_{c1} + B_{c1} y_1, \tag{4.21}$$
$$u_1 = C_{c1} x_{c1}, \tag{4.22}$$
$$\dot{x}_{c2} = A_{c2} x_{c2} + B_{c2} y_2, \tag{4.23}$$
$$u_2 = C_{c2} x_{c2}. \tag{4.24}$$

Letting $\mathcal{K}$ denote the block-diagonal matrix

$$\mathcal{K} = \text{block-diag}(A_{c1}, B_{c1}, C_{c1}, A_{c2}, B_{c2}, C_{c2}), \tag{4.25}$$

it can be verified that $L_{\mathcal{K}}$ is nonsingular. Hence, the closed-loop system consisting of (4.17)-(4.24) can be written as decentralized static output feedback

with $m = v = 6$, $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = \phi_6 = 1$, and $G(s)$ given by

$$G(s) \sim \left[\begin{array}{ccc|cccccc|c|c} A & 0 & 0 & 0 & 0 & B_1 & 0 & 0 & B_2 & 0 & D_1 \\ 0 & 0 & 0 & I & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 & 0 & F_{11} & 0 & 0 & F_{12} & 0 & D_{21} \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 & 0 & F_{21} & 0 & 0 & F_{22} & 0 & D_{22} \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline E_1 & 0 & 0 & 0 & 0 & E_{21} & 0 & 0 & E_{22} & 0 & E_0 \end{array}\right]. \tag{4.26}$$

4.4 Centralized Strictly Proper Dynamic Compensation with Normal Form Parameterization, Nominal Plant

Consider a centralized strictly proper dynamic compensator (4.14) with the dynamics matrix parameterized in normal form as

$$A_c = \text{block-diag}(A_{c1}, \ldots, A_{c\delta}), \tag{4.27}$$

where

$$A_{ci} = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}, \quad i = 1, \ldots, \delta, \tag{4.28}$$

and the matrices B_c and C_c remain unconstrained. The order of the compensator is thus 2δ. Defining

$$V_{ij} = \begin{bmatrix} 0_{2(i-1)\times 1} & 0_{j-1\times 1} & 1 & 0_{2-j\times 1} & 0_{2(\delta-i)\times 1} \end{bmatrix}^{\mathrm{T}}, \tag{4.29}$$

and letting $\mathcal{K}$ denote the block-diagonal matrix

$$\mathcal{K} = \text{block-diag}(a_1, a_1, b_1, b_1, \ldots, a_\delta, a_\delta, b_\delta, b_\delta, B_c, C_c), \tag{4.30}$$

it can be verified that $L_{\mathcal{K}}$ is nonsingular. Hence, the closed-loop system consisting of (4.1)-(4.3) and (4.13)-(4.14), with the parameterization (4.27), can be written as decentralized static output feedback with $m = 2\delta + 2$, $v = \delta + 2$,

$\phi_1 = \phi_2 = \cdots = \phi_\delta = 2$, $\phi_{\delta+1} = \phi_{\delta+2} = 1$, and $G(s)$ given by

$$
G(s) \sim \left[\begin{array}{cc|ccccccccccc|c|c}
A & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & B & 0 & D_1 \\
0 & 0 & V_{11} & V_{12} & V_{11} & -V_{12} & \cdots & V_{\delta 1} & V_{\delta 2} & V_{\delta 1} & -V_{\delta 2} & I & 0 & 0 & 0 \\
\hline
0 & V_{11}^{\mathrm{T}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_{12}^{\mathrm{T}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_{12}^{\mathrm{T}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_{11}^{\mathrm{T}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & V_{\delta 1}^{\mathrm{T}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_{\delta 2}^{\mathrm{T}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_{\delta 2}^{\mathrm{T}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_{\delta 1}^{\mathrm{T}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & F & 0 & D_2 \\
0 & I & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
E_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & E_2 & 0 & E_0
\end{array}\right]
\tag{4.31}
$$

4.5 Centralized Strictly Proper Dynamic Compensation with Uncertain Plant Dynamics

Consider the controller structure given by (4.13)–(4.14), and assume that uncertainty in the dynamics of the plant is accounted for by the model

$$
\dot{x} = (A + M_A \Delta_A N_A)x + Bu + D_1 w. \tag{4.32}
$$

Letting K be given by (4.15) and letting $\Delta = \Delta_1 = \Delta_A$, the closed-loop system consisting of (4.2), (4.3), (4.13)–(4.14), and (4.32) can be written as decentralized static output feedback with $m = r = 3$, $\phi_1 = \phi_2 = \phi_3 = 1$, $p = 1$, $\psi_1 = 1$, and $G(s)$ given by

$$
G(s) \sim \left[\begin{array}{cc|cc|c|c|c}
A & 0 & 0 & 0 & B & M_A & D_1 \\
0 & 0 & I & I & 0 & 0 & 0 \\
\hline
0 & I & 0 & 0 & 0 & 0 & 0 \\
C & 0 & 0 & 0 & F & 0 & D_2 \\
0 & I & 0 & 0 & 0 & 0 & 0 \\
\hline
N_A & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
E_1 & 0 & 0 & 0 & E_2 & 0 & E_0
\end{array}\right]. \tag{4.33}
$$

4.6 Centralized Strictly Proper Dynamic Compensation with Uncertain Input and Output Matrices

Consider the controller structure given by (4.13)–(4.14), and let the uncertainty in the input and output matrices of the plant be represented by

$$\dot{x} = Ax + (B + M_B \Delta_B N_B)u + D_1 w \tag{4.34}$$

and

$$y = (C + M_C \Delta_C N_C)x + Fu + D_2 w. \tag{4.35}$$

Letting $\mathcal{K}$ be defined by (4.15) and defining Δ as

$$\Delta = \text{block-diag}(\Delta_B, \Delta_C),$$

the closed-loop system consisting of (4.3), (4.13), (4.14), (4.34), and (4.35) can be written as decentralized static output feedback with $m = v = 3$, $\phi_1 = \phi_2 = \phi_3 = 1$, $p = 2$, $\psi_1 = \psi_2 = 1$, and $G(s)$ given by

$$G(s) \sim \left[\begin{array}{cc|ccc|cc|c} A & 0 & 0 & 0 & B & M_B & 0 & D_1 \\ 0 & 0 & I & I & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & F & 0 & M_C & D_2 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & N_B & 0 & 0 & 0 \\ N_C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline E_1 & 0 & 0 & 0 & E_2 & 0 & 0 & E_0 \end{array}\right]. \tag{4.36}$$

4.7 Multiple Model Compensation

Consider a strictly proper, centralized dynamic compensator

$$\dot{x}_c = A_c x_c + B_c y, \tag{4.37}$$
$$u = C_c x_c \tag{4.38}$$

which is required to stabilize and provide acceptable performance for two distinct plants, namely,

$$\dot{x}_1 = A_1 x + B_1 u + D_{11} w, \tag{4.39}$$
$$y_1 = C_1 x + F_1 u + D_{21} w, \tag{4.40}$$
$$z_1 = E_{11} x + E_{21} u + E_{01} w, \tag{4.41}$$

and

$$\dot{x}_2 = A_2 x + B_2 u + D_{12} w, \tag{4.42}$$
$$y_2 = C_2 x + F_2 u + D_{22} w, \tag{4.43}$$
$$z_2 = E_{12} x + E_{22} u + E_{02} w. \tag{4.44}$$

Such multiple models may be used to represent failure modes of the nominal plant, or represent the plant dynamics at various values of an uncertain parameter [1].

Letting $\mathcal{K}$ denote the partitioned matrix

$$\mathcal{K} = \begin{bmatrix} A_c & 0 & 0 & 0 & 0 & 0 \\ 0 & A_c & 0 & 0 & 0 & 0 \\ 0 & 0 & B_c & 0 & 0 & 0 \\ 0 & 0 & 0 & B_c & 0 & 0 \\ 0 & 0 & 0 & 0 & C_c & 0 \\ 0 & 0 & 0 & 0 & 0 & C_c \end{bmatrix}, \tag{4.45}$$

the closed-loop systems consisting of (4.39)-(4.44) and (4.37)-(4.38) can be written as decentralized static output feedback with $m = 6$, $v = 3$, $\phi_1 = \phi_2 = \phi_3 = 2$, and $G_a(s)$ given by

$$G_a(s) \sim \left[\begin{array}{cccc|c|cccccc} A_1 & 0 & 0 & 0 & D_{11} & 0 & 0 & 0 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & 0 & D_{12} & 0 & 0 & 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & I & 0 & 0 \\ \hline E_{11} & 0 & 0 & 0 & E_{01} & 0 & 0 & 0 & 0 & E_{21} & 0 \\ 0 & E_{12} & 0 & 0 & E_{02} & 0 & 0 & 0 & 0 & 0 & E_{22} \\ \hline 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 & D_{21} & 0 & 0 & F_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & D_{22} & 0 & 0 & 0 & F_2 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{4.46}$$

Chapter 5

Homotopy Theory

5.1 Probability-One Homotopy Methods

Homotopies are a traditional part of topology, and have found significant application in nonlinear functional analysis and differential geometry [11]. Homotopy methods are globally convergent, which distinguishes them from most iterative methods, which are only locally convergent. The general idea of homotopy methods is to make a continuous transformation from an initial problem, which can be solved trivially, to the target problem.

Following [12], the theoretical foundation of all probability-one globally convergent homotopy methods is given in the following differential geometry theorem

Definition 5.1 *Let $U \subset R^m$ and $V \subset R^p$ be open sets, and let $\rho : U \times [0,1) \times V \rightarrow R^p$ be a C^2 map. ρ is said to be transversal to zero if the Jacobian matrix $D\rho$ has full rank on $\rho^{-1}(0)$.*

Theorem 5.1 *If $\rho(a, \lambda, x)$ is transversal to zero, then for almost all $a \in U$ the map*

$$\rho_a(\lambda, x) = \rho(a, \lambda, x) \tag{5.1}$$

is also transversal to zero; i.e., with probability one the Jacobian matrix $D\rho_a(\lambda, x)$ has full rank on $\rho_a^{-1}(0)$.

The recipe for constructing a globally convergent homotopy algorithm to solve the nonlinear system of equations

$$f(x) = 0, \tag{5.2}$$

where $f : R^p \rightarrow R^p$ is a C^2 map, is as follows: For an open set $U \subset R^m$ construct a C^2 homotopy map $\rho : U \times [0,1) \times R^p \rightarrow R^p$ such that

1) $\rho(a, \lambda, x)$ is transversal to zero,
2) $\rho_a(0, x) = \rho(a, 0, x) = 0$ is trivial to solve and has a unique solution x_0,
3) $\rho_a(1, x) = f(x)$,
4) $\rho_a^{-1}(0)$ is bounded.

Then for almost all $a \in U$ there exists a zero curve γ of ρ_a, along which the Jacobian matrix $D\rho_a$ has rank p, emanating from $(0, x_0)$ and reaching a zero $\bar{x}$ of f at $\lambda = 1$. This zero curve γ does not intersect itself, is disjoint from any other zeros of ρ_a, and has finite arc length in every compact subset of $[0, 1) \times R^p$. Furthermore, if $Df(\bar{x})$ is nonsingular, then γ has finite arc length. The general idea of the algorithm is to follow the zero curve γ emanating from $(0, x_0)$ until a zero $\bar{x}$ of $f(x)$ is reached (at $\lambda = 1$).

The zero curve γ is tracked by the normal flow algorithm [12], a predictor-corrector sceme. In the predictor phase, the next point is produced using Hermite cubic interpolation. Starting at the predicted point, the corrector iteration involves computing (implicitly) the Moore-Penrose pseudo-inverse of the Jacobian matrix at each point. The most complex part of the homotopy algorithm is the computation of the tangent vectors to γ, which involves the computation of the kernel of the $p \times (p + 1)$ Jacobian matrix $D\rho_a$. The kernel is found by computing a QR factorization of $D\rho_a$, and then using back substitution. This strategy is implemented in the mathematical software package HOMPACK [14], which was used for the curve tracking here.

Two different homotopy maps are used for solving the optimal projection equations. When the initial problem, $g(x; a) = 0$, can be solved, then the homotopy map is [13]

$$\rho_a(\lambda, x) = F(a, \lambda, x) \equiv \lambda f(x) + (1 - \lambda) g(x; a), \tag{5.3}$$

where $f(x) = 0$ is the final problem, and a is a parameter vector used in defining the function g.

When the initial problem is not solved exactly, i.e., $g(x_0; b) \neq 0$, then the map is a Newton homotopy [9]

$$\rho_a(\lambda, x) = F(b, \lambda, x) - (1 - \lambda) F(b, 0, x_0), \tag{5.4}$$

where $a = (b, x_0)$. For $\lambda = 0$, $\rho_a(0, x_0) = F(b, 0, x_0) = 0$, and for $\lambda = 1$, $\rho_a(1, x) = F(b, 1, x) = f(x) = 0$.

When the map (5.4) is ised, the equations considered for $0 < \lambda < 1$ are not the optimal projection equations whenever $F(b, 0, x_0) = g(x_0; b) \neq 0$. Hence, a goal in constructing the initial system and the starting point may be to minimize $g(x_0; b)$.

Chapter 6

Formulation (Homotopy)

6.1 The $\mathcal{H}_2$-optimal order reduction problem

The $\mathcal{H}_2$-optimal model order reduction problem is that of approximating a higher order dynamical system by one of lower order so that a quadratic model reduction criterion is minimized.

The problem can be formulated as: given the asymptotically stable, controllable, observable, time invariant, continuous time system

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{6.1}$$

$$y(t) = C\,x(t), \tag{6.2}$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, $C \in \mathbf{R}^{l \times n}$, the goal is to find a reduced order model

$$\dot{x}_m(t) = A_m\,x_m(t) + B_m\,u(t), \tag{6.3}$$

$$y_m(t) = C_m\,x_m(t), \tag{6.4}$$

where $A_m \in \mathbf{R}^{n_m \times n_m}$, $B_m \in \mathbf{R}^{n_m \times m}$, $C_m \in \mathbf{R}^{l \times n_m}$, $n_m < n$, which minimizes the cost function

$$J(A_m, B_m, C_m) \equiv \lim_{t \to \infty} \mathcal{E}\left[(y - y_m)^T\, R\,(y - y_m)\right], \tag{6.5}$$

where the input $u(t)$ is white noise with symmetric and positive definite intensity V and R is a symmetric and positive definite weighting matrix.

6.2 The combined $\mathcal{H}_2/\mathcal{H}_\infty$ model reduction problem

In practice, to simplify a plant for control design or to simplify a controller for ease of implementation, a $\mathcal{H}_\infty$ role must be taken into account, i.e., the order

reduction approach should approximate the system frequency response to the greatest extent possible.

The problem is formulated as: given the controllable and observable, time invariant, continuous time system

$$\dot{x}(t) = A\,x(t) + B\,Du(t), \tag{6.6}$$
$$y(t) = C\,x(t), \tag{6.7}$$

where $t \in [0, \infty)$, $A \in \mathbf{R}^{n\times n}$ is asymptotically stable, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{ln}$, $D \in \mathbf{R}^{m\times p}$ $(m \leq p)$ and the input $Du(t)$ is white noise with symmetric and positive definite intensity $V \equiv DD^T$, find a n_m-th order model $(n_m < n)$

$$\dot{x}_m(t) = A_m\,x_m(t) + B_m\,Du(t), \tag{6.8}$$
$$y_m(t) = C_m\,x_m(t), \tag{6.9}$$

where $A_m \in \mathbf{R}^{n_m\times n_m}$, $B_m \in \mathbf{R}^{n_m\times m}$, $C_m \in \mathbf{R}^{ln_m}$, which satisfies the following criteria:
(i) A_m is asymptotically stable;
(ii) the transfer function of the reduced order model lies within γ of the transfer function of the full order model in the H_∞ norm, i.e.,

$$||H(s) - H_m(s)||_\infty \leq \gamma \tag{6.10}$$

where

$$H(s) \equiv EC(sI_n - A)^{-1}BD, \qquad H_m(s) \equiv EC_m(sI_m - A_m)^{-1}B_mD, \tag{6.11}$$

$\gamma > 0$ is a given constant, $E \in \mathbf{R}^{q\times l}$ $(q \geq l)$ is a given constant matrix; and
(iii) the H^2 model reduction criterion

$$J(A_m, B_m, C_m) \equiv \lim_{t\to\infty} \mathcal{E}\left[(y - y_m)^T R(y - y_m)\right] \tag{6.12}$$

is minimized, where $\mathcal{E}$ is the expected value and $R = E^TE$ is a symmetric and positive definite weighting matrix.

6.3 The LQG controller synthesis with an $\mathcal{H}_\infty$ performance bound

The $\mathcal{H}_2/\mathcal{H}_\infty$ mixed-norm controller synthesis problem provides the means for simultaneously addressing $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives. In practice such controllers provide both nominal performance (via $\mathcal{H}_2$) and robust stability (via $\mathcal{H}_\infty$). Hence mixed-norm synthesis provides a technique for trading off performance and robustness, a fundamental objective in control design. (It should be noted that $\mathcal{H}_2$ controller synthesis is a special case of mixed-norm controller synthesis, with the $\mathcal{H}_\infty$ bound set to ∞).

The LQG controller synthesis problem with an H^∞ performance bound can be stated as: given the n-th order stabilizable and detectable plant

$$\dot{x}(t) = A\,x(t) + B\,u(t) + D_1 w(t), \tag{6.13}$$
$$y(t) = C\,x(t) + D_2 w(t), \tag{6.14}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{l\times n}$, $D_1 \in \mathbf{R}^{n\times p}$, $D_2 \in \mathbf{R}^{l\times p}$, $D_1 D_2^T = 0$, and $w(t)$ is p-dimensional white noise, find a n_c-th order dynamic compensator

$$\dot{x}_c(t) = A_c\,x_c(t) + B_c\,y(t), \tag{6.15}$$
$$u(t) = C_c\,x_c(t), \tag{6.16}$$

where $A_c \in \mathbf{R}^{n_c\times n_c}$, $B_c \in \mathbf{R}^{n_c\times l}$, $C_c \in \mathbf{R}^{m\times n_c}$, and $n_c \le n$, which satisfies the following criteria:
(i) the closed-loop system (6.13) - (6.16) is asymptotically stable, i.e.,

$$\tilde{A} = \begin{pmatrix} A & BC_c \\ B_c C & A_c \end{pmatrix} \tag{6.17}$$

is asymptotically stable;
(ii) the $q_\infty \times p$ closed-loop transfer function from $w(t)$ to $E_{1\infty}x(t) + E_{2\infty}u(t)$,

$$H(s) \equiv \tilde{E}_\infty (sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D}, \tag{6.18}$$

where

$$\tilde{E}_\infty = \begin{pmatrix} E_{1\infty} & E_{2\infty}C_c \end{pmatrix} \left(E_{1\infty} \in \mathbf{R}^{q_\infty\times n},\ E_{2\infty} \in \mathbf{R}^{q_\infty\times m},\ E_{1\infty}^T E_{2\infty} = 0 \right), \tag{6.19}$$
$$\tilde{n} = n + n_c, \tag{6.20}$$
$$\tilde{D} = \begin{pmatrix} D_1 \\ B_c D_2 \end{pmatrix} \tag{6.21}$$

satisfy the constraint

$$\|H(s)\|_\infty \le \gamma \tag{6.22}$$

where $\gamma > 0$ is a given constant, and
(iii) the performance functional

$$J(A_c, B_c, C_c) \equiv \lim_{t\to\infty} \mathcal{E}\left[x^T(t)R_1 x(t) + u^T(t)R_2 u(t)\right] \tag{6.23}$$

is minimized, where $\mathcal{E}$ is the expected value, $R_1 = E_1^T E_1 \in \mathbf{R}^{n\times n}$ and $R_2 = E_2^T E_2 \in \mathbf{R}^{m\times m}$ ($E_1 \in \mathbf{R}^{q\times n}$, $E_2 \in \mathbf{R}^{q\times m}$, $E_1^T E_2 = 0$) are respectively symmetric positive semidefinite and symmetric positive definite weighting matrices.

6.4 Functions

The thirteen functions listed here are classified by their purpose. (They are described in detail, with examples of their use, in Chapter 7).

morh2inf, **morh2ly**, **morh2over**, and morh2op are for the $\mathcal{H}_2$-optimal model order reduction problem. **morh2hiinf**, **morh2hily**, and **morh2hiover** are for the combined $\mathcal{H}_2/\mathcal{H}_\infty$ model order reduction problem. For the reduced order LQG ($\mathcal{H}_2$-optimal) problem use **rlqgly**, **rlqginf**, or **rlqgover**. To solve the full-order LQG ($\mathcal{H}_2$-optimal) controller synthesis problem with an $\mathcal{H}_\infty$ norm bound, use **flqgly**, **flqginf**, or **flqgover**.

6.5 Final Note

For a complete explanation of the homotopy algorithms described above, see [6].

Chapter 7

Program Descriptions

dfolqg, drolqg

Purpose

To synthesize full- and reduced-order discrete-time $\mathcal{H}_2$-optimal dynamic compensators.

Synopsis

```
[Ac,Bc,Cc,Dc,cost] = dfolqg(A,Bu,Bw,Cy,Cz,Dyu,Dyw,Dzu,Dzw)
[Ac,Bc,Cc,Dc,test] = drolqg(A,Bu,Bw,Cy,Cz,Dyu,Dyw,Dzu,Dzw,nc)
```

Description

dfolqg synthesizes discrete-time $\mathcal{H}_2$-optimal compensators based on routines in the MATLAB CONTROL SYSTEMS TOOLBOX. Given an nth-order two-vector imput, two-vector output plant

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_u u(k) + B_w w(k), \\
y(k) &= C_y x(k) + D_{yu} u(k) + D_{yw} w(k), \\
z(k) &= C_z x(k) + D_{zu} u(k) + D_{zw} w(k),
\end{aligned}
$$

dfolqg designs a full-order $\mathcal{H}_2$-optimal strictly-proper dynamic compensator, of the form

$$
\begin{aligned}
x_c(k+1) &= A_c x_c(k) + B_c y(k) \\
u(k) &= C_c x_c(k) + D_c y(k).
\end{aligned}
$$

Note that the realization A_c, B_c, C_c, D_c returned by **dfolqg** is <u>not</u> the same as that produced by **dreg** in the CONTROL SYSTEMS TOOLBOX, which uses the *current* estimator of [5], while **dfolqg** uses the *predictor* estimator. See the CONTROL SYSTEMS TOOLBOX User's Guide for details [7]. **cost** is the value of the discrete-time $\mathcal{H}_2$ norm for the closed-loop transfer function $\tilde{G}_{zw}(z)$.

drolqg uses a similarity transformation to balance the realization of the controller, and then eliminates $n - n_c$ of the least observable and controllable states in this realization in order to obtain a reduced-order approximation of the full-order compensator. Balanced realizations do not exist for unstable controllers, and **drolqg** will fail with an error message if **dfolqg** returns an unstable compensator. **test** is a flag which signals whether the balanced truncation returned by **drolqg** stabilizes the closed-loop system.

test	$= 0 \rightarrow$	closed-loop stable,
test	$= 1 \rightarrow$	closed-loop unstable.

Notes

See **dlqr** and **dlqe** from CONTROL SYSTEMS TOOLBOX.

Examples

```
>> A = [1.0000 0.0001 0 0; -0.000025 1.0000 0 0; 0 0 1.0 0.0001;
0 0 -0.0004 1.0000];
>> B = [0, 0; 0.00005, 0.00005; 0, 0; 0.0001, 0.00020];
>> D1 = [B, zeros(4,2)];
>> C = [0 1 0 -0.5;0 1 0 -1.0];
>> E1 = [C; 0 0 0 0; 0 0 0 0];
>> D = zeros(2,2);
>> D2 = [0, 0, 0.1, 0; 0, 0, 0, 0.1];
>> E2 = [0, 0; 0, 0; 0.1, 0; 0, 0.1];
>> E0 = zeros(4,4);
>> who
Your variables are:
A          C          D1         E0         E2
B          D          D2         E1
>> [Ac,Bc,Cc,Dc,cost] = dfolqg(A,B,D1,C,E1,D,D2,E2,E0)
Ac =
    1.0000   -0.0003         0    0.0002
    0.0001    1.0004   -0.0004   -0.0009
         0   -0.0004    1.0000    0.0003
    0.0002    0.0037   -0.0012    0.9958
Bc =
    0.0003    0.0001
    0.0004   -0.0006
    0.0004   -0.0000
   -0.0001   -0.0021
Cc =
    1.5604   -7.6711   -8.2375   -0.4167
    0.4664   11.4192    0.2584  -10.3769
Dc =
     0     0
     0     0
cost =
   2.3624e-05
```

```
>> [Ac,Bc,Cc,Dc,test] = drolqg(A,B,D1,C,E1,D,D2,E2,E0,2)
Closed-Loop System UNSTABLE
Ac =
    1.0000    0.0000
   -0.0000    1.0000
Bc =
    0.0005    0.0068
   -0.0341    0.0177
Cc =
    0.0005    0.0341
    0.0068   -0.0177
Dc =
     0     0
     0     0
test =
     1
```

dsofformat

Purpose

To transform standard control problem into the equivalent decentralized static output feedback framework problem.

Synopsis

```
dsofformat(filename,ncvec,noivec,A,Bu,Bw,Cy,
Cz,Dyu,Dyw,Dzu,Dzw)
```

```
dsofformat(filename,ncvec,noivec, A,Bu,Bw,Cy,
Cz,Dyu,Dyw,Dzu,Dzw,Ma,Na,Mb,Nb,Mc,Nc)
```

Description

dsofformat reformulates a k-channel fixed-structure control problems into the equivalent decentralized static output feedback framework problem.

ncvec is a $k \times 1$ vector whose i-th element is the order of the dynamics for the i-th processor. Static gain compensators are indicated by a dimension of 0.

noivec is a **k**×2 matrix. The i-th element of the first column contains the dimension of measurement signal for the i-th channel, and the i-th element of the second column contains the dimension of the actuator signal for the i-th channel

If the arguments **Ma**. **Na**, etc , are included, **dsofformat** will generate the matrices required to represent the following plant uncertainty in the decentralized static output feedback framework

$$G(s) \sim \left[\begin{array}{c|c} A + M_A \Delta_A N_A & B_u + M_B \Delta_B N_B \\ \hline C_y + M_C \Delta_C N_C & D_{yu} \end{array} \right] \qquad (7.1)$$

The output of **dsofformat** consists of the matrices of the decentralized static output feedback realization (2.2), the matrices Q_{Lij} and Q_{Rij} which define the structure of the controller (2.14), and the repetition and sizing variables **N**, **kindex**, **indexkc**, and **indexkr** (see **optgain** for details on these variables). These output variables are saved to the file *filename*.**mat**.

Notes

see also **optgain**

Examples

First we will demonstrate setting up single-channel, reduced-order centralized strictly-proper dynamic compensator problem:

```
>> clear
>> A = [zeros(5,1),eye(5);-1    -2   -24   -12   -24     -4];
>> B = [zeros(5,1);1];
>> C = [1 0 0 0 0 0];
>> D = 0;
>> D1 = [B,zeros(6,1)];
>> D2 = [0,1];
>> E1 = [C;zeros(1,6)];
>> E2 = [0;1];
>> E0 = zeros(2,2);
>> who
Your variables are:
A          C          D1         E0         E2
B          D          D2         E1
>> dsofformat('data',4,[1,1],A,B,D1,C,E1,D,D2,E2,E0);
>> clear
>> load data
>> who
Your variables are:
A          Cz         Dzw        QL31       indexkc
Bu         Dyu        N          QR11       indexkr
Bw         Dyw        QL11       QR21       kindex
Cy         Dzu        QL21       QR31
>>
```

The matrices contained in **data.mat** are the transformed matrices corresponding to the realization (2.2). The matrices **QL**ij and **QR**ij correspond to (2.14).

A second example shows an example of setting up a two-channel decentralized dynamic compensator synthesis problem in the decentralized static output feedback framework, including real-parameter uncertainty in the input matrix.

```
>> clear
>> A = [0, 1;-3 -4];
>> B = [0, 0;-1, -.3];
>> D1 = [35, 0, 0; -61, 0, 0];
>> C = [2, 1;3, 1];
```

```
>> E1 = [52.1950, 8.9440; 0, 0; 0, 0];
>> D = zeros(2,2);
>> D2 = [0, 1, 0; 0, 0, 1];
>> E2 = [0, 0; 1, 0; 0, 1];
>> E0 = zeros(3,3);
>> Mb = B;
>> Nb = eye(2);
>> dsofformat('datafile',[2;2],[1,1;1,1],A,B,D1,C,E1,D,D2,E2,E0,[],[],Mb,Nb,[],[]);
>> clear
>> load datafile
>> who
Your variables are:
A        Cz       Dyw      QL31     QR31     kindex
Bd       Ded      Dzu      QL41     QR41
Bu       Deu      Dzw      QL51     QR51
Bw       Dew      N        QL61     QR61
Ce       Dyd      QL11     QR11     indexkc
Cy       Dyu      QL21     QR21     indexkr
>>
```

For more examples on using **dsofformat**, see Chapter 4.

flqgly, flqginf, flqgover

Purpose

Find the full-order LQG compensator with an $\mathcal{H}_\infty$ norm bound.

Synopsis

```
[Ac, Bc, Cc, cost] = flqgly(A, B, C, D, gamma0, gamma, E1, E2, E1i,
E2i, D1, D2)
[Ac, Bc, Cc, cost] = flqginf(A, B, C, D, gamma0, gamma, E1, E2,
E1i, E2i, D1, D2 )
[Ac, Bc, Cc, cost] = flqgover(A, B, C, D, gamma0, gamma, E1, E2,
E1i, E2i, D1, D 2)
```

Description

For a given n^{th} order linear plant with open-loop state space realization given by

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + D_1 w(t) && (7.2)\\
y(t) &= Cx(t) + D_2 w(t) && (7.3)\\
z(t) &= E_1 x(t) + E_2 u(t) && (7.4)\\
z_\infty(t) &= E_{1\infty} x(t) + E_{2\infty} u(t) && (7.5)
\end{aligned}$$

flqgly, **flqginf**, and **flqgover** calculate A_c, B_c, and C_c, a state space realization for the full-order LQG ($\mathcal{H}_2$ optimal) compensator which yields a closed-loop system with $\mathcal{H}_\infty$ norm bounded by **gamma**. The closed-loop $\mathcal{H}_2$ cost is given by **cost**. **gamma0** (γ_0) is the initial - and should always be greater than **gamma**.

The resulting compensator from tt flqginf is in the input normal Riccati form while that from **flqgly** is in Ly's form.

Examples

```
>> A=zeros(8);
>> B=zeros(8,1);
>> C=zeros(1,8);
>> D=0;
>> A(1:8,1) = [-0.161; -6.004; -0.5822; -9.9835; -0.4073;  -3.982;
0;
0;];
>> for i =1:7 A(i,i+1) = 1; end
```

```
>> B(1:8,1) = [0; 0;  0.0064; 0.00235; 0.0713; 1.0002;0.1045; 0.9955;];
>> C(1,1) = 1.0;
>> E1 = zeros(2,8);
>> E1(1,1:8) = 0.001 * [0  0 0 0 0.55 11 1.32 18];
>> E1i = E1;
>> E2 = [0;1];
>> E2i = [0;0];
>> D1 = zeros(8,2);
>> D1(1:8,1) = B;
>> D2 = [0 1];
>> [Ac,Bc,Cc,cost] = flqgly(A,B,C,D,1.0e3,2.0,E1,E2,E1i,E2i,D1,D2)
Ac =
  Columns 1 through 7
    0.0000    1.0000   -0.0000    0.0000   -0.0000   -0.0000    0.0000
   -3.4454   -0.1189   -0.0000    0.0000   -0.0000    0.0000    0.0000
   -0.0000   -0.0000    0.0000    1.0000    0.0000   -0.0000   -0.0000
    0.0000    0.0000   -1.9890   -0.1688    0.0000    0.0000   -0.0000
    0.0000    0.0000    0.0000    0.0000   -0.0000    1.0000    0.0000
   -0.0000   -0.0000   -0.0000   -0.0000   -0.6728   -0.1541   -0.0000
    0.0000    0.0000         0   -0.0000    0.0000   -0.0000    0.0000
   -0.0000    0.0000    0.0000    0.0000   -0.0000    0.0000   -0.3032
  Column 8
    0.0000
    0.0000
   -0.0000
   -0.0000
    0.0000
   -0.0000
    1.0000
   -0.8793
Bc =
    0.0012
    0.0048
    0.0061
    0.0094
    0.0068
    0.0143
```

```
   -0.1278
    0.1000
Cc =
  Columns 1 through 7
    1.0000    0.0000    1.0000    0.0000    1.0000   -0.0000    1.0000
  Column 8
    0.0000
cost =
    0.1434
```

Algorithm

The algorithms for **flqgly**, **flqginf**, and **flqgover** are described in Chapters 14, 15, 16, 17, and 18 of [6].

See Also

rlqgly, **rlqginf**, **rlqgover**

folqg, rolqg

Purpose

To synthesize full and reduced-order continuous-time $\mathcal{H}_2$-optimal dynamic compensators.

Synopsis

```
[Ac,Bc,Cc,Dc,cost] = folqg(A,Bu,Bw,Cy,Cz,Dyu,Dyw,Dzu,Dzw)
[Ac,Bc,Cc,Dc,test] = rolqg(A,Bu,Bw,Cy,Cz,Dyu,Dyw,Dzu,Dzw,nc)
```

Description

folqg synthesizes continuous-time $\mathcal{H}_2$-optimal compensators based on routines in the MATLAB CONTROL SYSTEMS TOOLBOX. Given an nth-order two-vector input, two-vector output plant

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \\
y(t) &= C_y x(t) + D_{yu} u(t) + D_{yw} w(t), \\
z(t) &= C_z x(t) + D_{zu} u(t) + D_{zw} w(t),
\end{aligned}$$

folqg designs a full-order $\mathcal{H}_2$-optimal strictly-proper dynamic compensator, of the form

$$\begin{aligned}
\dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\
u(t) &= C_c x_c(t) + D_c y(t).
\end{aligned}$$

In standard LQG theory, $D_c = 0$. **cost** is the value of the $\mathcal{H}_2$ norm for the closed-loop transfer function $G_{zw}(s)$.

rolqg uses a similarity transformation to balance the realization of the controller, and then eliminates $n - n_c$ of the least observable and controllable states in this realization in order to obtain a reduced-order approximation of the full-order compensator. Balanced realizations do not exist for unstable controllers, and **rolqg** will fail with an error message if **folqg** returns an unstable compensator. **test** is a flag which signals whether the balanced truncation returned by **rolqg** stabilizes the closed-loop system.

test	$= 0 \rightarrow$	closed-loop stable,
test	$= 1 \rightarrow$	closed-loop unstable.

Notes

See **lqr**, **lqe**, and **reg** from CONTROL SYSTEMS TOOLBOX.

Examples

```
>> A = [zeros(5,1),eye(5);-1    -2   -24   -12   -24    -4];
>> B = [zeros(5,1);1];
>> C = [1 0 0 0 0 0];
>> D = 0;
>> D1 = [B,zeros(6,1)];
>> D2 = [0,1];
>> E1 = [C;zeros(1,6)];
>> E2 = [0;1];
>> E0 = zeros(2,2);
>> [Ac,Bc,Cc,Dc,cost] = folqg(A,B,D1,C,E1,D,D2,E2,E0)
Ac =
   -0.1480    1.0000         0         0         0         0
   -0.0110         0    1.0000         0         0         0
    0.0103         0         0    1.0000         0         0
    0.0016         0         0         0    1.0000         0
   -0.0041         0         0         0         0    1.0000
   -1.4149   -5.4340  -25.9960  -15.5854  -24.6029   -4.1480
Bc =
    0.1480
    0.0110
   -0.0103
   -0.0016
    0.0041
    0.0007
Cc =
   -0.4142   -3.4340   -1.9960   -3.5854   -0.6029   -0.1480
Dc =
     0
cost =
    0.2822
>> [Ac,Bc,Cc,Dc,test] = rolqg(A,B,D1,C,E1,D,D2,E2,E0,4)
Ac =
   -0.0157   -0.3137    0.0046   -0.0016
    0.3137   -0.3487    0.0275   -0.0093
   -0.0046    0.0275   -0.2634    1.0135
   -0.0016    0.0093   -1.0135   -0.0360
```

```
Bc =
   -0.0763
    0.2845
   -0.0112
   -0.0038
Cc =
   -0.0763   -0.2845    0.0112   -0.0038
Dc =
     0
test =
     0
```

morh2inf, morh2ly, morh2over, morh2op

Purpose

Find the $\mathcal{H}_2$ optimal reduced-order model of a linear system.

Synopsis

```
[Am, Bm, Cm, cost] = morh2inf(A, B, C, nm)
[Am, Bm, Cm, cost] = morh2ly(A, B, C, nm)
[Am, Bm, Cm, cost] = morh2over(A, B, C, nm)
[Am, Bm, Cm, cost] = morh2op(A, B, C, nm, meth, init, c1, c2)
```

Description

For a given linear system with state space representaion A, B, and C, **morh2inf**, **morh2ly**, **morh2over**, and **morh2op** return the $\mathcal{H}_2$ optimal reduced-order model A_m, B_m, and C_m of dimension nm with $\mathcal{H}_2$ cost **cost**. The result from **morh2inf** is in the input normal form while that from **morh2ly** is in Ly's form.

In **morh2op**, **meth** and **init** denote the strategy and the method of initialization respectively [15], and **c1** and **c2** define the initial A by $A_0 = -c1I + c2A$.

Examples

```
>> A = [-10 1 0; -5 0 1; -1 0 0];
>> B = [0 ; 1 ; 1];
>> C = [1  0  0];
>> nm = 2;
>> [Am, Bm, Cm, cost ] = morh2inf(A, B, C, nm)
Am =
   -0.1397   -0.1006
    0.6010   -0.4482
Bm =
   -0.5285
    0.9468
Cm =
   -0.3204   -0.0961
cost =
    3.2902e-04
>> [Am, Bm, Cm, cost ] = morh2ly(A, B, C, nm)
```

```
Am =

         0    1.0000

   -0.1231   -0.5878

Bm =

    0.0784

    0.0782

Cm =

    1.0000    0.0000

cost =

    3.2902e-04

>> A = [-1 3 0; -1 -1 1; 4 -5 -4];
>> B =[ -2; 2; 4];
>> C = [1 0 0];
>> nm = 1;
>> [Am, Bm, Cm, cost ] = morh2op(A, B, C, nm, 2, 2, 10.0, 0.0)

Am =

  -10.4365

Bm =

   -1.5972

Cm =

    1.5972

cost =

    1.6882
```

Algorithm

The algorithms for **morh2inf**, **morh2ly**, and **morh2over** are described in Chapters 2, 4, and 6 respectively of [6]. The algorithm for morh2op can be found in [15].

See Also

morh2hiinf, morh2hily, morh2hiover

morh2hiinf, morh2hily, morh2hiover

Purpose

Find the combined $\mathcal{H}_2/\mathcal{H}_\infty$ reduced-order model of a linear system.

Synopsis

```
[Am, Bm, Cm, cost] = morh2hiinf(A, B, C, nm, gamma)
[Am, Bm, Cm, cost] = morh2hily(A, B, C, nm, gamma)
[Am, Bm, Cm, cost] = morh2hiover(A, B, C, nm, gamma)
```

Description

For a given linear plant with state space representaion A, B, and C, **morh2hiinf**, **morh2hily**, and **morh2hiover** return the combined $\mathcal{H}_2/\mathcal{H}_\infty$ reduced-order model A_m, B_m, and C_m of dimension nm with $\mathcal{H}_2$ cost **cost**. The triple (A_m, B_m, C_m) returned from **morh2hiinf** is in the input normal form while that from **morh2hily** is in Ly's form.

Examples

```
>> A = zeros(10);
>> B = zeros(10, 1);
>> C = zeros(1,10);
>> A(1,1:10) = [-10 -45 -120 -210 -252 -210 -120 -45 -10 -1];
>> for i=1:9 A(i+1,i)= 1.0; end
>> B(1,1) = 1.0;
>> C(1,10) = 1.0;
>> nm = 4;
>> gamma = 1.0;
>> [Am, Bm, Cm, cost ]= morh2hiinf(A, B, C, nm, gamma)
Am =
   -0.0273   -0.1286   -0.0274    0.0124
    0.2376   -0.1087   -0.1936    0.0397
   -0.1352    0.5178   -0.2416    0.2322
   -0.2412    0.4166   -0.9124   -0.4787
Bm =
    0.2338
   -0.4663
```

```
    0.6951
    0.9785

Cm =

    0.1897    0.2047    0.1142   -0.0409

cost =

    1.2868e-04

>> [Am, Bm, Cm, cost ]= morh2hiover(A, B, C, nm, gamma)

Am =

   -0.0308   -0.1739   -0.0642    0.0566
    0.1739   -0.1200   -0.3132    0.1338
   -0.0642    0.3132   -0.2566    0.4553
   -0.0566    0.1338   -0.4553   -0.4489

Bm =

    0.2148
   -0.3180
    0.2916
    0.2044

Cm =

    0.2148    0.3180    0.2916   -0.2044

cost =

    1.2868e-04
```

Algorithm

The algorithms for **morh2hiinf**, **morh2hily**, and **morh2hiover** are described in Chapters 8, 9, and 10 respectively of [6].

See Also

morh2op, **morh2inf**, **morh2ly**, **morh2over**

optgain

Purpose

To optimize fixed-structure controllers using a quasi-Newton gradient search algorithm.

Synopsis

```
[fmin, info, noit] = optgain(infile, outfile)
[fmin, info, noit] = optgain(infile, outfile, outon, diag,
maxit, method, tolfac)
```

Description

optgain optimizes fixed-structure controllers using a quasi-newton gradient search algorithm. The data defining the problem to be solved is contained in the file *infile*`.mat`, and must include the following:

- `A,Bu,Bw,Cy,Cz,Dyu,Dyw,Dzu,Dzw`, the matrices of the decentralized static output feedback framework realization (2.2).
- `QL`ij ,`QR`ij, $i = 1,\ldots,\iota$, $j = 1,\ldots,o(i)$, the matrices defining the structure of $\mathcal{K}$ (2.14)
- **kindex**, defined as

$$\texttt{kindex} \stackrel{\Delta}{=} \begin{bmatrix} o_1 & o_2 & \cdots & o_\iota \end{bmatrix}, \qquad (7.6)$$

  where the $o(i)$ are defined by (2.12).
- **indexkc** and **indexkr**, defined as

$$\texttt{indexkc} \stackrel{\Delta}{=} \begin{bmatrix} c_1 & c_2 & \cdots & c_\iota \end{bmatrix}, \qquad (7.7)$$

$$\texttt{indexkr} \stackrel{\Delta}{=} \begin{bmatrix} r_1 & r_2 & \cdots & r_\iota \end{bmatrix}, \qquad (7.8)$$

  where c_i and r_i are defined by (2.12).
- **ctype**, which defines the performance criterion which will be used

ctype	Performance Criterion
1	Continuous-time $\mathcal{H}_2$
3	scaled Popov
-1	Discrete-Time $\mathcal{H}_2$

- `k1, k2, ...k`ι, initial values for the parameter matrices to be optimized (2.8)

- N, defined as $N \triangleq v$

In addition, scaled Popov Synthesis requires the following variables also be defined:

- **Bd,Dyd,Ce,Deu,Ded,Dew,Dzd**, the matrices of the decentralized static output feedback realization (2.2) defining the model uncertainty.
- **WW** and **ZZ**, initial values for the scaling and stability multiplier matrices (3.12),
- **blk**, a matrix which defines the block structure of the uncertainty matrix Δ.

The optimized parameters of the controller are written to the file *outfile*.**mat**.

The optional arguments of **optgain** are defineds as follows:

- **outon** ≥ 1 is an integer which determines how often during the iteration process output will be displayed to the screen (and saved to the diagnostic files, if chosen; see below). Screen displays are updated every **outon** iterates. The default value for **outon** is 1.
- **diag** $= 0/1$ is a flag for turning the *diagnostics recorder* on or off. The diagnostic recorder saves such information as gradient norm vs. iteration no., cost function vs. iteration no., etc. If **diag** $= 1$, the diagnostic recorder is on, and **optgain** will generate **.mat** files with this data. If **diag** $= 0$, the recorder is off. The default value for **diag** is 0.
- **maxit** ≥ 1 is an integer which represents the maximum allowable number of iterations. The default value for **maxit** is 500.
- **method** is a flag variable which determines which search algorithm the quasi-Newton optimization code will use to search for the next candidate parameter vector, as follows: (For more

method	Search Algorithm
1	Line Search
2	Double Dog-Leg Search
3	"Hook" Step

information on these methods, see [4]). The default value for **method** is 1.
- **tolfac**> 1 is a multiplication factor for the code-supplied tolerances within the quasi-Newton algorithm, which are on the order of the machine epsilon. By increasing **tolfac**, the user

can relax the tolerances for the optimization stopping criteria. The default value for `tolfac` is 1.

`optgain` returns as output `fmin`, the minimum value of the cost function it achieves, the termination code from the optimization routine, `info`, and `noits` the total number of iterations performed. `info` is interpereted as follows,

info	Reason for Termination
0	Optimal Solution Found
1	Small Gradient
2	Small Step Length
3	Unable to Find Lower Cost
4	Iteration Limit `maxit` Exceeded
5	Too Many Large Steps (Unbounded Cost Function)
-1	Insufficient Workspace

Examples

Here we set up a reduced-order, continous-time $\mathcal{H}_2$-optimal control problem, using a centralized, strictly-proper dynamic compensator.

```
>> A = [zeros(5,1),eye(5);-1   -2   -24   -12   -24    -4];
>> B = [zeros(5,1);1];
>> C = [1 0 0 0 0 0];
>> D = 0;
>> D1 = [B,zeros(6,1)];
>> D2 = [0,1];
>> E1 = [C;zeros(1,6)];
>> E2 = [0;1];
>> E0 = zeros(2,2);
>> dsofformat('data',1,[1,1],A,B,D1,C,E1,D,D2,E2,E0);
>> [k1,k2,k3,k4,test] = rolqg(A,B,D1,C,E1,D,D2,E2,E0,1);
>> clear k4
>> save init k1 k2 k3
>> clear
>> ctype = 1;
>> load data
>> load init
>> who
```

```
Your variables are:
A          Cz         Dzw        QL31       indexkc    k3
Bu         Dyu        N          QR11       indexkr    kindex
Bw         Dyw        QL11       QR21       k1
Cy         Dzu        QL21       QR31       k2
>> save run1
>> clear
>> [fmin, info, noits] = optgain('run1','outfile')
 USING LINE SEARCH ALGORITHM
     ITERATE             FUNCTION VALUE
     -------             --------------
         0            0.4106206586956E+00
         1            0.4055239785049E+00
         2            0.3553510422910E+00
         3            0.3550592319726E+00
         4            0.3538665020519E+00
         5            0.3532465844085E+00
         :            :
        24            0.3157312171398E+00
        25            0.3153669369866E+00
        26            0.3153121011808E+00
        27            0.3153090709009E+00
        28            0.3153090381912E+00
 TOTAL ITERATIONS =     28
 UNCMND WARNING -- INFO = 1: PROBABLY CONVERGED, GRADIENT SMALL
fmin =
    0.3153
info =
     1
noits =
    28
>>
```

pv_lmi

Purpose

To provide initializing stability multiplier and scaling matrices for scaled Popov synthesis.

Synopsis

```
[WW,ZZ,fail] = pv_lmi(A,B,C,R,gamma,blk)
```

Description

pv_lmi finds a stability multiplier matrix **ZZ** and scaling matrix **WW** for use with continuous-time scaled Popov criterion performance optimization problems. These matrices correspond to the solution of the scaled Popov Riccati equation (3.13) as the solution to an optimization problem subject to several linear matrix inequality (LMI) constraints [2]. The optimization problem is defined as

$$\min \operatorname{tr} P$$

subject to

$$\begin{bmatrix} (A+\gamma^{-1}BC)^{\mathrm{T}}P + P(A+\gamma^{-1}BC) + R & PB + C^{\mathrm{T}}Z + (A+\gamma^{-1}BC)^{\mathrm{T}}C^{\mathrm{T}}W \\ B^{\mathrm{T}}P + ZC + WC(A+\gamma^{-1}BC) & WCB + B^{\mathrm{T}}C^{\mathrm{T}}W - \gamma Z \end{bmatrix} < 0$$

$$P > 0$$
$$Z > 0$$

fail is a flag which is returned 0 if **pv_lmi** was successful in finding a feasible pair of matrices, or 1 otherwise. **ZZ** and **WW** are returned as **Null** if the LMI solver is unsuccessful.

rlqgly, rlqginf, rlqgover

Purpose

Find the reduced-order LQG compensator with an $\mathcal{H}_\infty$ bound.

Synopsis

```
[Ac, Bc, Cc, cost] = rlqgly(A,B,C,D, nc, gamma0, gamma, beta, E1,
E2, E1i, E2i, D1, D2)
[Ac, Bc, Cc, cost] = rlqginf(A,B,C,D, nc, gamma0, gamma, beta, E1,
E2, E1i, E2i, D1, D2)
[Ac, Bc, Cc, cost] = rlqgover(A,B,C,D, nc, gamma0, gamma, beta,
E1, E2, E1i, E2i , D1, D2)
```

Description

For a given nth order linear plant with open-loop state space realization given by

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + D_1 w(t) && (7.9)\\
y(t) &= Cx(t) + D_2 w(t) && (7.10)\\
z(t) &= E_1 x(t) + E_2 u(t) && (7.11)\\
z_\infty(t) &= E_{1\infty} x(t) + E_{2\infty} u(t) && (7.12)
\end{aligned}$$

rlqgly, **rlqginf**, and **rlqgover** calculate A_c, B_c, and C_c, a state space realization for the LQG ($\mathcal{H}_2$ optimal) **nc**th order compensator which yields a closed-loop system with $\mathcal{H}_\infty$ norm bounded by **gamma**. The closed-loop $\mathcal{H}_2$ cost is given by **cost**. **gamma0** (γ_0) is the initial γ and should always be greater than **gamma**. **beta** ($\beta \gg 0$) is a positive number. The resulting compensator from **rlqginf** is in the input normal Riccati form while that from **rlqgly** is in Ly's form.

Examples

```
>> A=zeros(8);
>> B=zeros(8,1);
>> C=zeros(1,8);
>> D=0;
>> A(1:8,1) = [-0.161; -6.004;-0.5822; -9.9835; -0.4073;  -3.982;
0; 0;];
>> for i =1:7 A(i,i+1) = 1; end
>> B(1:8,1) = [0; 0;  0.0064; 0.00235; 0.0713; 1.0002;0.1045; 0.9955;];
>> C(1,1) = 1.0;
```

```
>> E1 = zeros(2,8);
>> E1(1,1:8) = 0.001 * [0  0 0 0 0.55 11 1.32 18];
>> E1i = E1;
>> E2 = [0;1];
>> E2i = [0;0];
>> D1 = zeros(8,2);
>> D1(1:8,1) = B;
>> D2 = [0 1];
>> [Ac,Bc,Cc,cost] = rlqgly(A,B,C,D,2,1.0e3,3.8,100,E1,E2,E1i,E2i,D1,D2)
Ac =
          0    1.0000
    -0.0965   -0.2452
Bc =
    -0.1968
    -0.1410
Cc =
     1.0000   -0.0000
cost =
     2.8821
```

Algorithm

The algorithms for **rlqgly**, **rlqginf**, and **rlqgover** are described in Chapters 14, 15, 16, 17, and 18 of [6]

See Also

flqgly, **flqginf**, **flqgover**

siso2tito, tito2siso

Purpose

transform between single- and dual-vector input/output formats

Synopsis

```
[A,Bu,Bw,Cy,Cz,Dyu,Dyw,Dzu,Dzw] = siso2tito(A,B,C,D,nu,ny)
[A,B,C,D] = tito2siso(A,Bu,Bw,Cy,Cz,Dyu,Dyw,Dzu,Dzw)
```

Description

`siso2tito` and `tito2siso` transform continuous- and discrete-time state-space realizations between single-vector input/single-vector output (SVISVO) and two-vector input/two-vector output (TVITVO) format. Given the TVITVO plant

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \\
y(t) &= C_y x(t) + D_{yu} u(t) + D_{yw} w(t), \\
z(t) &= C_z x(t) + D_{zu} u(t) + D_{zw} w(t),
\end{aligned} \tag{7.13}
$$

where $D_{yu} \in \mathcal{R}^{n_y \times n_u}$. `tito2siso` concatenates the signals u and w to return the SVISVO plant

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t),
\end{aligned} \tag{7.14}
$$

where $u^{\mathrm{T}} = \begin{bmatrix} u^{\mathrm{T}} w^{\mathrm{T}} \end{bmatrix}$, and

$$
\begin{aligned}
B &= \begin{bmatrix} B_u & B_w \end{bmatrix} \\
C &= \begin{bmatrix} C_y \\ C_z \end{bmatrix} \\
D &= \begin{bmatrix} D_{yu} & D_{yw} \\ D_{zu} & D_{zw} \end{bmatrix}
\end{aligned}
$$

Conversely, given the system (7.14), `siso2tito` will return the TVITVO plant (7.13) by selecting the first n_u columns of B as the matrix B_u, the first n_y rows of C as C_y, and breaking up D accordingly.

Examples

```
>> A = [zeros(5,1),eye(5);-1    -2   -24   -12   -24    -4];
>> B = [zeros(5,1);1];
>> C = [1 0 0 0 0 0];
>> D = 0;
>> D1 = [B;zeros(6,1)];
>> D2 = [0,1];
>> E1 = [C;zeros(1,6)];
>> E2 = [0;1];
>> E0 = zeros(2,2);
>> [Abar,Bbar,Cbar,Dbar] = tito2siso(A,B,D1,C,E1,D,D2,E2,E0)
Abar =
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
    -1    -2   -24   -12   -24    -4
Bbar =
     0     0     0
     0     0     0
     0     0     0
     0     0     0
     0     0     0
     1     1     0
Cbar =
     1     0     0     0     0     0
     1     0     0     0     0     0
     0     0     0     0     0     0
Dbar =
     0     0     1
     0     0     0
     1     0     0
>> [A,B,D1,C,E1,D,D2,E2,E0] = siso2tito(Abar,Bbar,Cbar,Dbar,1,1)
A =
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
```

```
     0     0     0     0     1     0
     0     0     0     0     0     1
    -1    -2   -24   -12   -24    -4
B =
     0
     0
     0
     0
     0
     1
D1 =
     0     0
     0     0
     0     0
     0     0
     0     0
     1     0
C =
     1     0     0     0     0     0
E1 =
     1     0     0     0     0     0
     0     0     0     0     0     0
D =
     0
D2 =
     0     1
E2 =
     0
     1
E0 =
     0     0
     0     0
```

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